NEHRU COLLEGE OF ENGINEERING AND RESEARCH CENTRE
(NAAC Accredited)
(Approved by AICTE, Affiliated to APJ Abdul Kalam Technological University, Kerala)

DEPARTMENT OF ELECTRONICS & COMMUNICATION ENGINEERING

COURSE MATERIALS

ECT 303: DIGTAL SIGNAL PROCESSING

VISION OF THE INSTITUTION

To mould true citizens who are millennium leaders and catalysts of change through excellence in education.

MISSION OF THE INSTITUTION

NCERC is committed to transform itself into a center of excellence in Learning and Research in Engineering and Frontier Technology and to impart quality education to mould technically competent citizens with moral integrity, social commitment and ethical values.

We intend to facilitate our students to assimilate the latest technological know-how and to imbibe discipline, culture and spiritually, and to mould them in to technological giants, dedicated research scientists and intellectual leaders of the country who can spread the beams of light and happiness among the poor and the underprivileged.

ABOUT DEPARTMENT

- Established in: 2002
- Course offered : B.Tech in Electronics and Communication Engineering

  M.Tech in VLSI
- Approved by AICTE New Delhi and Accredited by NAAC
- Affiliated to the University of Dr. A P J Abdul Kalam Technological University.

DEPARTMENT VISION

Providing Universal Communicative Electronics Engineers with corporate and social relevance towards sustainable developments through quality education.

DEPARTMENT MISSION

1) Imparting Quality education by providing excellent teaching, learning environment.

2) Transforming and adopting students in this knowledgeable era, where the electronic gadgets (things) are getting obsolete in short span.

3) To initiate multi-disciplinary activities to students at earliest and apply in their respective fields of interest later.

4) Promoting leading edge Research & Development through collaboration with academia & industry.

PROGRAMME EDUCATIONAL OBJECTIVES

PEOI. To prepare students to excel in postgraduate programmes or to succeed in industry / technical profession through global, rigorous education and prepare the students to practice and innovate recent fields in the specified program/ industry environment.

PEO2. To provide students with a solid foundation in mathematical, Scientific and engineering fundamentals required to solve engineering problems and to have strong practical knowledge required to design and test the system.

PEO3. To train students with good scientific and engineering breadth so as to comprehend, analyze, design, and create novel products and solutions for the real life problems.

PEO4. To provide student with an academic environment aware of excellence, effective communication skills, leadership, multidisciplinary approach, written ethical codes and the life-long learning needed for a successful professional career.

PROGRAM OUTCOMES (POS)

Engineering Graduates will be able to:

1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.
2. **Problem analysis**: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design/development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and team work**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one’s own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

12. **Life-long learning**: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

PROGRAM SPECIFIC OUTCOMES (PSO)

PSO1: Ability to Formulate and Simulate Innovative Ideas to provide software solutions for Real-time Problems and to investigate for its future scope.

PSO2: Ability to learn and apply various methodologies for facilitating development of high quality
System Software Tools and Efficient Web Design Models with a focus on performance optimization.

PSO3: Ability to inculcate the Knowledge for developing Codes and integrating hardware/software
products in the domains of Big Data Analytics, Web Applications and Mobile Apps to create innovative career path and for the socially relevant issues.

COURSE OUTCOMES
EC 405

SUBJECT CODE: EC 308	
COURSE OUTCOMES	
C303.1	State and prove the fundamental properties and relations relevant to DFT and solve basic problems involving DFT based filtering methods
C303.2	Compute DFT and IDFT using DIT and DIF radix-2 FFT algorithms
C303.3	Design linear phase FIR filters and IIR filters for a given specification
C303.4	Illustrate the various FIR and IIR filter structures for the realization of the given system function, Decimation and interpolation in both time and frequency domains
C303.5	Explain the architecture of DSP processor (TMS320C67xx) and the finite word length effects

MAPPING OF COURSE OUTCOMES WITH PROGRAM OUTCOMES

CO'S	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
C303.1	3	3	3	3	3	3			2	1		
C303.2	3	3	3	3	3	3			2	1		
C303.3	3	3	3	3	3	3			2	1		
C303.4	3	3	3	3	3	3			2	1		
C303.5	3	3	3	3	3	3			2	1		
C303	3	3	3	3	3	3			2	1		

CO'S	PSO1	PSO2	PSO3
C303.1	3	3	1
C303.2	3	3	1
C303.3	3	3	1
C303.4	3	3	1
C303.5	3	3	1
C303	3	3	1

SYLLABUS

SYLLABUS

Module 1

Basic Elements of a DSP system, Typical DSP applications, Finite-length discrete transforms, Orthogonal transforms – The Discrete Fourier Transform: DFT as a linear transformation (Matrix relations), Relationship of the DFT to other transforms, IDFT, Properties of DFT and examples. Circular convolution, Linear Filtering methods based on the DFT, linear convolution using circular convolution, Filtering of long data sequences, overlap save and overlap add methods, Frequency Analysis of Signals using the DFT (concept only required)

Module 2

Efficient Computation of DFT: Fast Fourier Transform Algorithms-Radix-2 Decimation in Time and Decimation in Frequency FFT Algorithms, IDFT computation using Radix-2 FFT Algorithms, Application of FFT Algorithms, Efficient computation of DFT of Two Real Sequences and a 2N-Point Real Sequence

Module 3

Design of FIR Filters - Symmetric and Anti-symmetric FIR Filters, Design of linear phase FIR filters using Window methods, (rectangular, Hamming and Hanning) and frequency sampling method, Comparison of design methods for Linear Phase FIR Filters. Design of IIRDigital Filters from Analog Filters (Butterworth), IIR Filter Design by Impulse Invariance, and Bilinear Transformation, Frequency Transformations in the Analog and DigitalDomain.

Module 4

Structures for the realization of Discrete Time Systems - Block diagram and signal flow graph representations of filters, FIR Filter Structures: Linear structures, Direct Form, CascadeForm, IIR Filter Structures: Direct Form, Transposed Form, Cascade Form and Parallel Form, Computational Complexity of Digital filter structures. Multi-rate Digital Signal Processing:

Decimation and Interpolation (Time domain and Frequency Domain Interpretation), Anti- aliasing and anti-imaging filter.

Module 5

Computer architecture for signal processing: Harvard Architecture, pipelining, MAC, Introduction to TMS320C67xx digital signal processor, Functional Block Diagram.
Finite word length effects in DSP systems: Introduction (analysis not required), fixed-point and floating-point DSP arithmetic, ADC quantization noise, Finite word length effects in IIRdigital filters: coefficient quantization errors. Finite word length effects in FFT algorithms: Round off errors

Text Books

1. Proakis J. G. and Manolakis D. G., Digital Signal Processing, 4/e, Pearson Education, 2007
2. Alan V Oppenheim, Ronald W. Schafer ,Discrete-Time Signal Processing, 3rd Edition , Pearson ,2010

3. Mitra S. K., Digital Signal Processing: A Computer Based Approach, 4/e McGraw Hill (India) 2014

Reference Books

4. Ifeachor E.C. and Jervis B. W., Digital Signal Processing: A Practical Approach, 2/e Pearson Education, 2009.
5. Lyons, Richard G., Understanding Digital Signal Processing, 3/e. Pearson Education India, 2004.
6. Salivahanan S, Digital Signal Processing,4e, Mc Graw –Hill Education New Delhi, 2019
7. Chassaing, Rulph., DSP applications using C and the TMS320C6x DSK. Vol. 13. John Wiley & Sons, 2003.
8. Vinay.K.Ingle, John.G.Proakis, Digital Signal Processing: Bookware Companion Series,Thomson,2004
9. Chen, C.T., "Digital Signal Processing: Spectral Computation & Filter Design", Oxford Univ. Press, 2001.
10. Monson H Hayes, "Schaums outline: Digital Signal Processing", McGraw HillProfessional, 1999

Course Contents and Lecture Schedule

No.	Topic	No. of Lectures
1	**Module 1**	
1.1	Basic Elements of a DSP system, Typical DSP applications, Finite length Discrete transforms, Orthogonal transforms	1
1.2	The Discrete Fourier Transform: DFT as a linear transformation(Matrix relations),	1
1.3	Relationship of the DFT to other transforms, IDFT	1
1.4	Properties of DFT and examples ,Circular convolution	2
1.5	Linear Filtering methods based on the DFT- linear convolution using circular convolution, Filtering of long data sequences, overlap save and overlap add methods,	3
1.6	Frequency Analysis of Signals using the DFT(concept only required)	1
2	**Module 2**	
2.1	Efficient Computation of DFT: Fast Fourier Transform Algorithms	1
2.2	Radix-2 Decimation in Time and Decimation in Frequency FFT Algorithms	4
2.3	IDFT computation using Radix-2 FFT Algorithms	2
2.4	Application of FFT Algorithms-Efficient computation of DFT of Two Real Sequences and a 2N-Point Real Sequence	1
3	**Module 3**	
3.1	Design of FIR Filters- Symmetric and Anti-symmetric FIR Filters, Design of linear phase FIR filters using Window methods, (rectangular, Hamming and Hanning)	4
3.2	Design of linear phase FIR filters using frequency sampling Method, Comparison of Design Methods for Linear Phase FIR Filters	2
3.3	Design of IIR Digital Filters from Analog Filters, (Butterworth), IIR Filter Design by Impulse Invariance	3
3.4	IIR Filter Design by Bilinear Transformation	2
3.5	Frequency Transformations in the Analog and Digital Domain.	1

4	**Module 4**	
4.1	Structures for the realization of Discrete Time Systems- Block diagram and signal flow graph representations of filters	2
4.2	FIR Filter Structures: (Linear structures), Direct Form Cascade Form	,2
4.3	IIR Filter Structures: Direct Form, Cascade Form and Parallel Form	3
4.3	Computational Complexity of Digital filter structures.	1
4.4	Multi-rate Digital Signal Processing: Decimation and Interpolation (Time domain and Frequency Domain Interpretation), Anti-aliasing and anti-imaging filter.	3
5	**Module 5**	
5.1	Computer architecture for signal processing : Harvard Architecture, pipelining, MAC, Introduction to TMS320C67xx digital signal processor ,Functional Block Diagram	3
5.2	Finite word length effects in DSP systems: Introduction (analysis not required), fixed-point and floating-point DSP arithmetic, ADC quantization noise,	3
5.3	Finite word length effects in IIR digital filters: coefficient quantization errors.	2
5.4	Finite word length effects in FFT algorithms: Round off errors	1

Mark distribution

Total Marks	CIE	ESE	ESE Duration
150	50	100	3 hours

Continuous Internal Evaluation Pattern:

Attendance : 10 marks
Continuous Assessment Test (2 numbers) : 25 marks
Assignment/Quiz/Course project : 15 marks

End Semester Examination Pattern: There will be two parts; Part A and Part B. Part A contain 10 questions with 2 questions from each module, having 3 marks for each question. Students should answer all questions. Part B contains 2 questions from each module of which student should answer any one. Each question can have maximum 2 sub-divisions and carry 14 marks.

QUESTION BANK

MODULE I				
Q:NO:	**QUESTIONS**	**CO**	**KL**	**PAGE NO:**
1	Derive the relationship of DFT to Fourier transform	CO1	K3	4
2	Explain the following properties of DFT a) Circular Convolution b) Time Reversal	CO1	K2	8
3	Derive the relationship of DFT to Z-transform.	CO1	K3	12
4	Explain the following properties of DFT a) Complex conjugate property b) Circular Convolution	CO1	K2	16
5	Explain the following properties of DFT a) Linearity b) Complex conjugate property	CO1	K2	23
6	Find the circular convolution of $x1\ (n)$ = {1, −1, −2,3, −1},$x2\ (n)$ = {1,2,3} Using i) Concentric circle method ii) Matrix metho	CO1	K3	30
7	Find the output y(n) of a filter whose impulse response is h(n)={1,1,1} and input x(n)={3,-1,0,1,3,2,0,1,2,1} using i) Overlap-save method ii) Overlap-add method	CO1	K3	31
8	Find the output y(n) of a filter whose impulse response is h(n)={1,1,1} and input x(n)={3,-1,0,1,3,2,0,1,2,1} using i) Overlap-save method ii) Overlap-add method K3/CO1 O	CO1	K3	33
9	The first eight points of 14-point DFT of a real valued sequence are {12, - 1+j3, 3+j4, 1-j5, -2+j2, 6+j3, -2-j3, 10,…..} i) Determine the remaining points ii) Evaluate x[0] without computing the IDFT of X(k)? iii) Evaluate IDFT to obtain the real sequence ?	CO1	K3	35
10	Find the remaining samples of the 14-point DFT of the sequence given below X(K)={12,-1+j3,3+j4,1-j5,-2+j2,6+j3,-2-j3,10,…..}	CO1	K3	37
11	Consider the sequence x(n)={1,2,-3,0,1,-1,4,2}. Evaluate the following functions without computing the DFT. i) X(0) ii) X(4) iii) $\sum X(K)$ 7 k=0 iv) $\sum e$ $-j3\pi k$ 4 7 k=0 X(K)	CO1	K3	40

MODULE II				
1	Find the IDFT of the sequence X(k)={10,-2+j2,-2,-2-j2} using DIT algorithm	CO2	K5	49
2	Compute 4-point DFT of a sequence x(n)={0,1,2,3} using DIF algorithm	CO2	K5	51
3	Compute 4-point DFT of a sequence x(n)={0,1,2,3} using DIT algorithm.	CO2	K5	52
4	Compute 4-point DFT of a sequence x(n)={1,0,0,1} using DIF algorithm	CO2	K5	53
5	Compute 4-point DFT of a sequence x(n)={0,1,2,3} using DIF algorithm.	CO2	K5	54
6	Compute 4-point DFT of a sequence x(n)={1,-1,1,-1} using DIT algorithm.	CO2	K5	62
7	Find the 8 point DFT of a real sequence x(n)={1,2,3,4,4,3,2,1} using radix-2 decimation in time algorithm	CO2	K3	65
8	Compute the eight point DFT of the sequence $x(n) = \{\ 1\ 0 \leq n \leq 7\ 0\ otherwise\}$ By using DIF algorithms.	CO2	K3	66
9	Compute the 8 point DFT of x(n) = {2,1,-1,3,5,2,4,1} using radix-2 decimation in time FFT algorithm.	CO2	K3	68
10	Find the 8 point DFT of a real sequence x(n)={1,2,2,2,1,0,0,0,0} using radix-2 decimation in frequency algorithm.	CO2	K3	70
11	Compute the eight point DFT of the sequence $x(n) = \{\ 1\ 0 \leq n \leq 7\ 0\ otherwise\}$ By using DIT algorithm	CO2	K3	72
MODULE III				
1	Illustrate the design of IIR filters from Analog Filters.	CO3	K3	99
2	Design an analog butterworth filter that has a -2dB passband attenuation at a frequency of 20rad/sec and atleast -10dB stopband attenuation at 30rad/sec.	CO3	K3	102
3	For the given specifications design an analog	CO3	K3	103

	Butterworth filter. 0.9≤ $\lvert H(j\Omega)\rvert \le 1$ for $0 \le \Omega \le 0.2\pi$. $\lvert H(j\Omega)\rvert \le 0.2$ for $0.4\pi \le \Omega \le \pi$.			
4	Design an analog butterworth filter that has a -2dB passband attenuation at a frequency of 20rad/sec and atleast -10dB stopband attenuation at 30rad/sec.	CO3	K3	105
5	Design an analog Butterworth filter that has a -2dB passband attenuation at a frequency of 20rad/sec and atleast -10dB stopband attenuation at 30rad/sec.	CO3	K3	107
6	Illustrate the design of IIR filters from Analog Filters.	CO3	K3	110
7	Design a digital butterworth filter satisfying the constraints 0.707 ≤ |H(e jω)| ≤ 1 for 0 ≤ ω ≤ π 2 |H(e jω)| ≤ 0.2 for 3π 4 ≤ ω ≤ π With T = 1 sec. Use Bilinear transform.	CO3	K6	111
8	Convert the analog filter H(s) given below into a 2 nd order butterworth digital filter using impulse invariance technique. H(s) = 1 s 2+√2 s+1	CO3	K3	113
9	Apply bilinear transformation to H(s)= 2 (s+1)(s+2) with T=1 sec and find H(Z)	CO3	K3	114
10	Design a digital butterworth filter satisfying the constraints 0.707 ≤ |H(e jω)| ≤ 1 for 0 ≤ ω ≤ π 2 |H(e jω)| ≤ 0.2 for 3π 4 ≤ ω ≤ π With T = 1 sec. Use Bilinear transform.	CO3	K6	116
	MODULE IV			
1	Define a signal flow graph. Draw the signal flow graph of first order digital filter.	CO4	K2	126
2	Sketch a cascade realization of FIR filter structure with complex zeros.	CO4	K3	135
3	Realize the transposed form structure for the system Y(n) = -0.1y(n-1) +0.2y(n-2) +3x(n)+ 3.6x(n-1) +0.6x(n-2)	CO4	K3	138
4	Realize the system with difference equation y(n)= 3 4 y(n − 1) − 1 8 y(n − 2) + x(n) + 1 3 x(n − 1) in cascade form.	CO4	K3	141
5	Draw the direct form I and direct form II structures for the difference equation y(n) = x(n)+0.5x(n-1) +3y(n-1)-2y(n-2)	CO4	K3	143
6	Draw the cascade form structure for a discrete time sequence described H(Z)= 1+ 1 2 z −1 1− 3 4 z−1+ 1 8 z−2	CO4	K3	146
7	Realize the system function using minimum number of multipliers H(Z) = (1+z −1)(1+0.5z −1 + 0.5z −2 + z −3)	CO4	K3	147

8	Obtain the parallel form structure for the system given by the difference equation y(n)= -0.1y(n-1) +0.72y(n-2) +0.7x(n)- 0.252x(n-2)	CO4	K3	149
9	Let a signal $x(n)$ = 0.5 □$u(n)$ is decimated by 2. What happens to its spectrum?	CO4	K2	162
10	Derive Decimation In Time (DIT) FFT algorithm for 8 point DFT and draw the signal flow graph.	CO4	K5	170
11	Explain the effect in the spectrum of a signal x(n) when it is (i) Decimated by a factor 3 (ii) Interpolated by a factor 2 (5)	CO4	K3	171
	MODULE V			
1	Draw the block diagram of TMS320C67xx and briefly explain function of all blocks.	CO5	K3	151
2	Draw the block diagram of ADC quantization noise and explain in detail.	CO5	K3	153
3	Explain the effects of coefficient quantization in FIR and IIR filters.	CO5	K2	154
4	Derive the variance of quantization noise in ADC. Assume step size is Δ.	CO5	K3	156
5	Let $x(n)$ = 0.5 □$u(n)$. Obtain the signals for decimation by 3, interpolation by 3.	CO5	K2	157
6	Write notes on finite word length effects in DSP systems.	CO5	K2	158

APPENDIX 1

CONTENT BEYOND THE SYLLABUS

S:NO;	TOPIC	PAGE NO:
1	Array Signal Processing	180

MODULE 1

1.1 INTRODUCTION

Signals constitute an important part of our daily life. Anything that carries some information is called a signal. A signal is defined as a single-valued function of one or more independent variables which contain some information. A signal is also defined as a physical quantity that varies with time, space or any other independent variable. A signal may be represented in time domain or frequency domain. Human speech is a familiar example of a signal. Electric current and voltage are also examples of signals. A signal can be a function of one or more independent variables. A signal may be a function of time, temperature, position, pressure, distance etc. If a signal depends on only one independent variable, it is called a one- dimensional signal, and if a signal depends on two independent variables, it is called a two- dimensional signal.

A system is defined as an entity that acts on an input signal and transforms it into an output signal. A system is also defined as a set of elements or fundamental blocks which are connected together and produces an output in response to an input signal. It is a cause-and- effect relation between two or more signals. The actual physical structure of the system determines the exact relation between the input $x(n)$ and the output $y(n)$, and specifies the output for every input. Systems may be single-input and single-output systems or multi-input and multi-output systems.

Signal processing is a method of extracting information from the signal which in turn depends on the type of signal and the nature of information it carries. Thus signal processing is concerned with representing signals in the mathematical terms and extracting information by carrying out algorithmic operations on the signal. Digital signal processing has many advantages over analog signal processing. Some of these are as follows:

Digital circuits do not depend on precise values of digital signals for their operation. Digital circuits are less sensitive to changes in component values. They are also less sensitive to variations in temperature, ageing and other external parameters.

In a digital processor, the signals and system coefficients are represented as binary words. This enables one to choose any accuracy by increasing or decreasing the number of bits in the binary word.

Digital processing of a signal facilitates the sharing of a single processor among a number of signals by time sharing. This reduces the processing cost per signal.

Digital implementation of a system allows easy adjustment of the processor characteristics during processing.

Linear phase characteristics can be achieved only with digital filters. Also, multirate processing is possible only in the digital domain. Digital circuits can be connected in cascade without any loading problems, whereas this cannot be easily done with analog circuits.

Storage of digital data is very easy. Signals can be stored on various storage media such as magnetic tapes, disks and optical disks without any loss. On the other hand, stored analog signals deteriorate rapidly as time progresses and cannot be recovered in their original form.

Digital processing is more suited for processing very low frequency signals such as seismic signals.

Though the advantages are many, there are some drawbacks associated with processing a signal in digital domain. Digital processing needs ‘pre’ and ‘post’ processing devices like analog-to- digital and digital-to-analog converters and associated reconstruction filters. This increases the complexity of the digital system. Also, digital techniques suffer from frequency limitations. Digital systems are constructed using active devices which consume power whereas analog processing algorithms can be implemented using passive devices which do not consume power. Moreover, active devices are less reliable than passive components. But the advantages of digital processing techniques outweigh the disadvantages in many applications. Also the cost of DSP hardware is decreasing continuously. Consequently, the applications of digital signal processing are increasing rapidly.

The digital signal processor may be a large programmable digital computer or a small microprocessor programmed to perform the desired operations on the input signal. It may also be a hardwired digital processor configured to perform a specified set of operations on the input signal.

DSP has many applications. Some of them are: Speech processing, Communication, Biomedical, Consumer electronics, Seismology and Image processing.

The block diagram of a DSP system is shown in Figure 1.1.

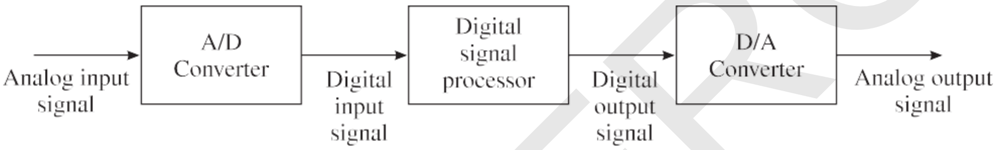


Figure 1.1 Block diagram of a digital signal processing system.

In this book we discuss only about discrete one-dimensional signals and consider only single- input and single-output discrete-time systems. In this chapter, we discuss about various basic discrete-time signals available, various operations on discrete-time signals and classification of discrete-time signals and discrete-time systems.

1.2REPRESENTATION OF DISCRETE-TIME SIGNALS

Discrete-time signals are signals which are defined only at discrete instants of time. For those signals, the amplitude between the two time instants is just not defined. For discrete-time signal the independent variable is time *n*, and it is represented by x (n).

There are following four ways of representing discrete-time signals:

1. Graphical representation
2. Functional representation
3. Tabular representation
4. Sequence representation

1.2.1 Graphical Representation

Consider a single x (n) with values

X (-2) = -3, x(-1) = 2, x(0) = 0, x(1) = 3, x(2) = 1 and x(3) = 2

This discrete-time single can be represented graphically as shown in Figure 1.2

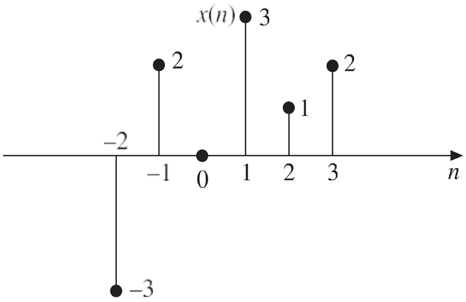


Figure 1.2 Graphical representation of discrete-time signal

1.2.1 Functional Representation

In this, the amplitude of the signal is written against the values of *n*. The signal given in section 1.2.1 can be represented using the functional representation as follows:

$$x(n)=\begin{cases} -3 & \text{for } n=-2 \\ 2 & \text{for } n=-1 \\ 0 & \text{for } n=0 \\ 3 & \text{for } n=1 \\ 1 & \text{for } n=2 \\ 2 & \text{for } n=3 \end{cases}$$

Another example is:

$$X(n) = 2^n u(n)$$

Or $$x(n) = \begin{cases} 2^n & for\ n \geq 0 \\ 0 & for\ n < 0 \end{cases}$$

1.2.3 Tabular Representation

In this, the sampling instant *n* and the magnitude of the signal at the sampling instant are representedin the tabular form. The signal given in section 1.2.1 can be represented in tabular form as follows:

n	2	1	0	1	2	3
x (*n*)	3	2	0	3	1	2

1.2.4 Sequence Representation

A finite duration sequence given in section 1.2.1 can be represented as follows:

$$X(n) = \{-3,2,0,3,1,2\}$$
$$\uparrow$$

Another example is:

$$X(n) = \{\ldots 2,3,0,1,-2 \ldots\}$$
$$\uparrow$$

The arrow mark ↑ denotes the *n* = 0 term. When no arrow is indicated, the first term correspondsto *n* = 0.

So a finite duration sequence, that satisfies the condition *x*(*n*) = 0 for *n* < 0 can be represented as:

$$x(n) = \{3, 5, 2, 1, 4, 7\}$$

Sum and product of discrete-time sequences

The sum of two discrete-time sequences is obtained by adding the corresponding elements ofsequences

$$\{C_n\} = \{a_n\} + \{b_n\} \rightarrow C_n = a_n + b_n$$

The product of two discrete-time sequences is obtained by multiplying the correspondingelements of the sequences.

$$\{C_n\} = \{a_n\}\{b_n\} \rightarrow C_n = a_n b_n$$

The multiplication of a sequence by a constant *k* is obtained by multiplying each element of thesequence by that constant.

$$\{C_n\} = k\{a_n\} \rightarrow C_n = ka_n$$

1.3 ELEMENTARY DISCRETE-TIME SIGNALS

There are several elementary signals which play vital role in the study of signals and systems. These elementary signals serve as basic building blocks for the construction of more complex signals. Infact, these elementary signals may be used to model a large number of physical signals, which occur in nature. These elementary signals are also called standard signals.

The standard discrete-time signals are as follows:

1. Unit step sequence
2. Unit ramp sequence
3. Unit parabolic sequence
4. Unit impulse sequence
5. Sinusoidal sequence
6. Real exponential sequence
7. Complex exponential sequence

1.3.1 Unit Step Sequence

The step sequence is an important signal used for analysis of many discrete-time systems. It exists only for positive time and is zero for negative time. It is equivalent to applying a signal whose amplitude suddenly changes and remains constant at the sampling instants forever after application. In between the discrete instants it is zero. If a step function has unity magnitude, then it is called unit step function.

The usefulness of the unit-step function lies in the fact that if we want a sequence to start at $n = 0$, so that it may have a value of zero for $n < 0$, we only need to multiply the given sequence with unit step function $u(n)$.

The discrete-time unit step sequence $u(n)$ is defined as:

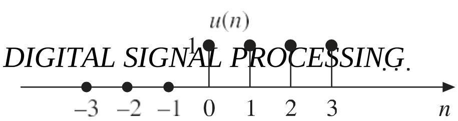
$$U(n) = \begin{cases} 1 & for\ n \geq 0 \\ 0 & for\ n < 0 \end{cases}$$

The shifted version of the discrete-time unit step sequence $u(n - k)$ is defined as:

$$U(n-k) = \begin{cases} 1 & for\ n \geq k \\ 0 & for\ n < k \end{cases}$$

It is zero if the argument $(n - k) < 0$ and equal to 1 if the argument $(n - k)$ S 0.

The graphical representation of $u(n)$ and $u(n - k)$ is shown in Figure 1.3[(a) and (b)].



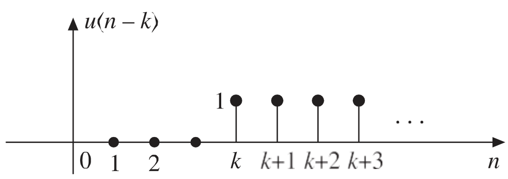


Figure 1.3 Discrete-time (a) Unit step function (b) Shifted unit step function

1.3.2 Unit Ramp Sequence

The discrete-time unit ramp sequence *r* (*n*) is that sequence which starts at *n* = 0 and increaseslinearly with time and is defined as:

$$r(n) = \begin{cases} n & for\ n \geq 0 \\ 0 & for\ n < 0 \end{cases}$$

or $$r(n) = nu(n)$$

It starts at *n* = 0 and increases linearly with *n*.

The shifted version of the discrete-time unit ramp sequence *r*(*n* – *k*) is defined as:

$$R(n - k) = \begin{cases} n - k & for\ n \geq k \\ 0 & for\ n < k \end{cases}$$

Or $$r(n - k) = (n - k)\ u(n - k)$$

The graphical representation of *r*(*n*) and *r*(*n* – 2) is shown in Figure 1.4[(a) and (b)].

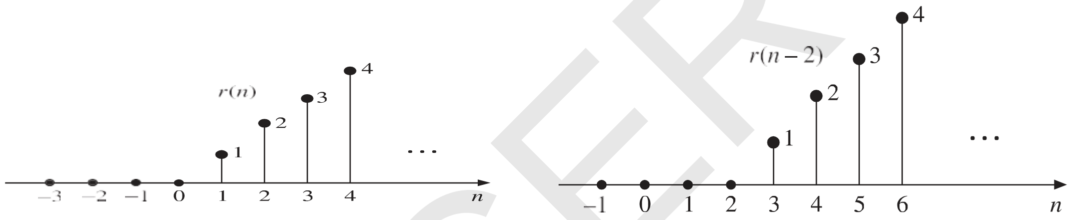


Figure 1.4 Discrete-time (a) Unit ramp sequence (b) Shifted ramp sequence.

1.3.3 Unit Parabolic Sequence

The discrete-time unit parabolic sequence *p* (*n*) is defined as:

$$P(n) = \begin{cases} \frac{n^2}{2} & for\ n \geq 0 \\ 0 & for\ n < 0 \end{cases}$$

Or $$P(n) = \frac{N^2}{2}\ u(n)$$

The shifted version of the discrete-time unit parabolic sequence *p*(*n* – *k*) is defined as:

$$P(n - k) = \begin{cases} \frac{(n-k)^2}{2} & for\ n \geq k \\ 0 & for\ n < k \end{cases}$$

Or $$p(n - k) = \frac{(n-k)^2}{2} u(n - k)$$

The graphical representation of *p*(*n*) and *p*(*n* – 3) is shown in Figure 1.5[(a) and (b)].

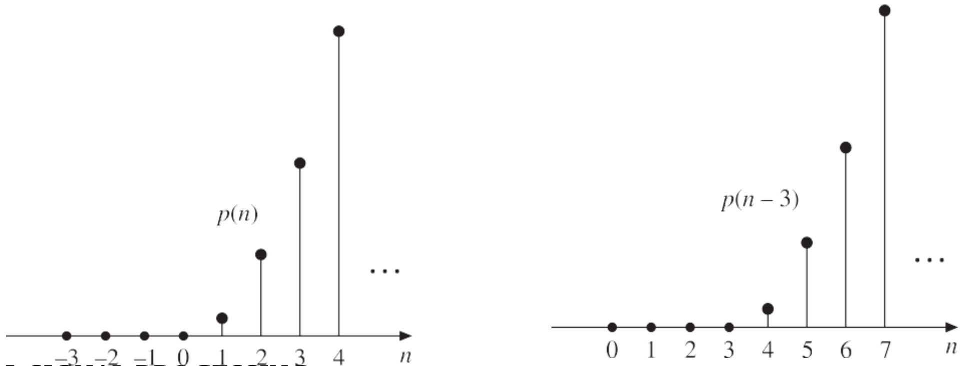


Figure 1.5 Discrete-time (a) Parabolic sequence (b) Shifted parabolic sequence.

1.3.4 Unit Impulse Function or Unit Sample Sequence

The discrete-time unit impulse function (*n*), also called unit sample sequence, is defined as:

$$\delta\,(n) = \begin{cases} 1 \; for \; n = 0 \\ 0 \; for \; n \neq 0 \end{cases}$$

This means that the unit sample sequence is a signal that is zero everywhere, except at *n* = 0, where its value is unity. It is the most widely used elementary signal used for the analysis of signals and systems.

The shifted unit impulse function (*n* – *k*) is defined as:

$$\delta\,(n-k) = \begin{cases} 1 \; for \; n = k \\ 0 \; for \; n \neq k \end{cases}$$

The graphical representation of (*n*) and (*n* – *k*) is shown in Figure 1.6[(a) and (b)].

Figure 1.6 Discrete-time (a) Unit sample sequence (b) Delayed unit sample sequence.

Properties of discrete-time unit sample sequence

1. δ(n) = u(n) – u(n – 1)
2. δ(n – k) = $\begin{cases} 1 \; for \; n = k \\ 0 \; for \; n \neq k \end{cases}$
3. X(n) = $\sum_{k=-\infty}^{\infty} x(k)\delta\,(n-k)$
4. $\sum_{n=-\infty}^{\infty} x(n)\delta\,(n-n_0)$ = x(n0)

Relation Between The Unit Sample Sequence And The Unit Step Sequence

The unit sample sequence $\delta(n)$ and the unit step sequence $u(n)$ are related as:

$$U(n) = \sum_{m=0}^{n} \delta\,(m),\ \delta(n) = u(n) - u(n-1)$$

Sinusoidal Sequence

The discrete-time sinusoidal sequence is given by

$$X(n) = A \sin\,(\omega n + \emptyset)$$

Where *A* is the amplitude, is angular frequency, is phase angle in radians and *n* is an integer.

The period of the discrete-time sinusoidal sequence is:

$$N = \frac{2\pi}{\omega} m$$

Where *N* and *m* are integers.

All continuous-time sinusoidal signals are periodic, but discrete-time sinusoidal sequences mayor may not be periodic depending on the value of.

For a discrete-time signal to be periodic, the angular frequency must be a rational multiple of 2. The graphical representation of a discrete-time sinusoidal signal is shown in Figure 1.7.

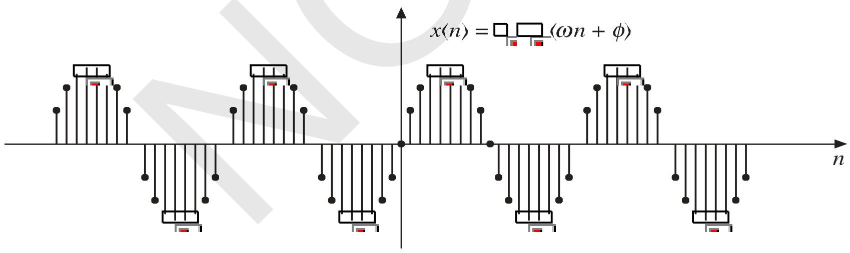


Figure 1.7 Discrete-time sinusoidal signal

1.3.6 Real Exponential Sequence

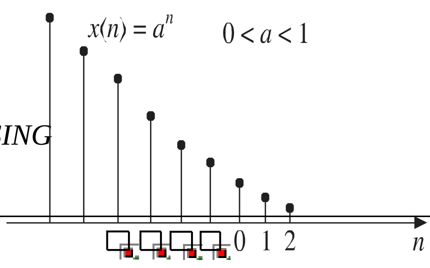
The discrete-time real exponential sequence a^n is defined as:

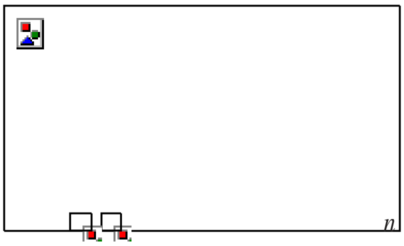
$$X(n) = a^n \text{ for all n}$$

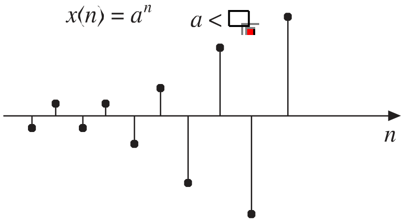
Figure 1.8 illustrates different types of discrete-time exponential signals.

When *a* > 1, the sequence grows exponentially as shown in Figure 1.8(a).

When 0 < *a* < 1, the sequence decays exponentially as shown in Figure 1.8(b). When *a* < 0, the sequence takes alternating signs as shown in Figure 1.8[(c) and (d)].







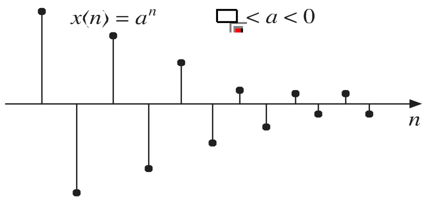


Figure 1.8 Discrete-time exponential signal an for (a) a > 1 (b) 0 < a < 1 (c) a < -1 (d) -1 < a < 0.

1.3.7 Complex Exponential Sequence

The discrete-time complex exponential sequence is defined as:

$$X(n) = a^n e^{j(\omega_0 n + \emptyset)}$$

$$= an\cos(\omega_0 n + \emptyset) + ja^n \sin(\omega_0 n = \emptyset)$$

For $|a| = 1$, the real and imaginary parts of complex exponential sequence are sinusoidal.

For $|a| > 1$, the amplitude of the sinusoidal sequence exponentially grows as shown inFigure 1.9(a).

For $|a| < 1$, the amplitude of the sinusoidal sequence exponentially decays as shown inFigure 1.9(b).

1.4BASIC OPERATIONS ON SEQUENCES

When we process a sequence, this sequence may undergo several manipulations involving theindependent variable or the amplitude of the signal.

The basic operations on sequences are as follows:

1. Time shifting
2. Time reversal
3. Time scaling
4. Amplitude scaling
5. Signal addition
6. Signal multiplication

The first three operations correspond to transformation in independent variable *n* of a signal. The last three operations correspond to transformation on amplitude of a signal.

1.4.1 Time Shifting

The time shifting of a signal may result in time delay or time advance. The time shifting operationof a discrete-time signal $x(n)$ can be represented by

$$y(n) = x(n - k)$$

This shows that the signal $y(n)$ can be obtained by time shifting the signal $x(n)$ by k units. If k is positive, it is delay and the shift is to the right, and if k is negative, it is advance and the shift is to the left.

An arbitrary signal $x(n)$ is shown in Figure 1.10(a). $x(n – 3)$ which is obtained by shifting $x(n)$ to the right by 3 units (i.e. delay $x(n)$ by 3 units) is shown in Figure 1.10(b). $x(n + 2)$ whichis obtained by shifting $x(n)$ to the left by 2 units (i.e. advancing $x(n)$ by 2 units) is shown in

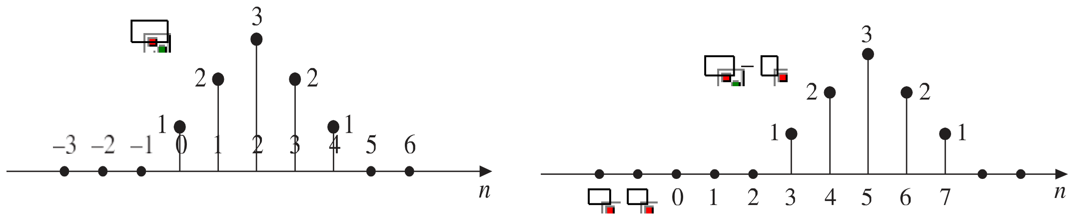


Figure 1.10(c).

Figure 1.10 (a) Sequence x(n) (b) x(n – 3) (c) x(n + 2).

1.4.2 Time Reversal

The time reversal also called time folding of a discrete-time signal *x*(*n*) can be obtained by foldingthe sequence about *n* = 0. The time reversed signal is the reflection of the original signal. It is obtained by replacing the independent variable *n* by –*n*. Figure 1.11(a) shows an arbitrary discrete-time signal *x*(*n*), and its time reversed version *x*(–*n*) is shown in Figure 1.11(b).

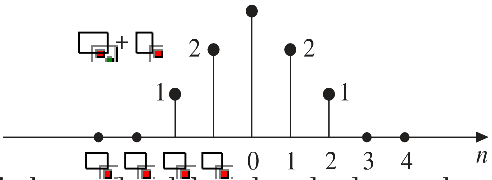


Figure 1.11[(c) and (d)] shows the delayed and advanced versions of reversed signal *x*(–*n*).

The signal *x*(–*n* + 3) is obtained by delaying (shifting to the right) the time reversed signal *x*(–*n*) by 3 units of time. The signal *x*(–*n* – 3) is obtained by advancing (shifting to theleft) the time reversed signal *x*(–*n*) by 3 units of time.

Figure 1.12 shows other examples for time reversal of signals

EXAMPLE 1.2 Sketch the following signals:

(a) U(n+2) u(-n+3) (b) x(n) = u(n+4) – u(n-2)

Solutions:

(a) Given x(n)=u(n+2) u(-n+3)

The signal *u* (*n* + 2) *u*(–*n* + 3) can be obtained by first drawing the signal *u*(*n* + 2) as shown inFigure 1.13(a), then drawing *u*(–*n* + 3) as shown in Figure 1.13(b),

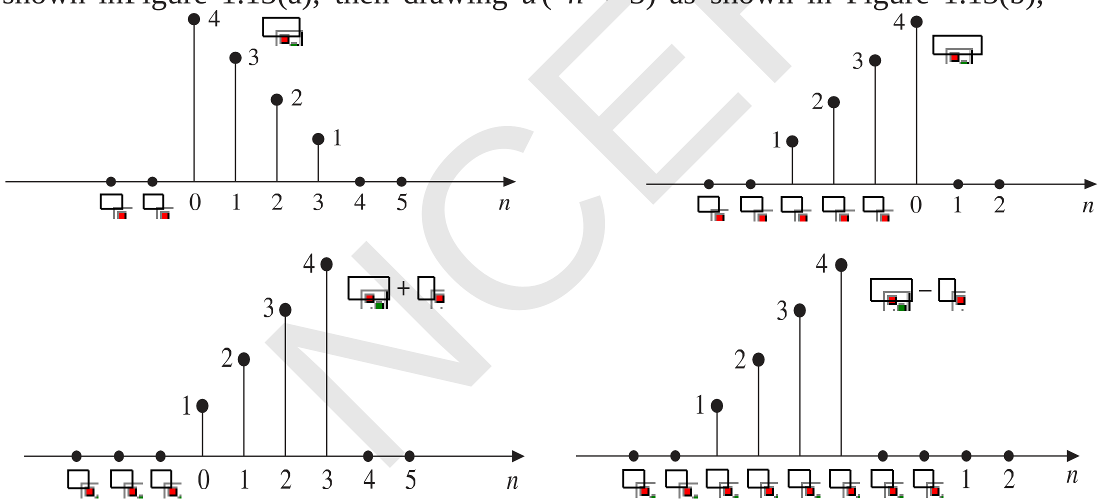


Figure 1.11 (a) Original signal x(n) (b) Time reversed signal x(-n) (c) Time reversed anddelayed

signal x(-n+3) (d) Time reversed and advanced signal x(-n-3).

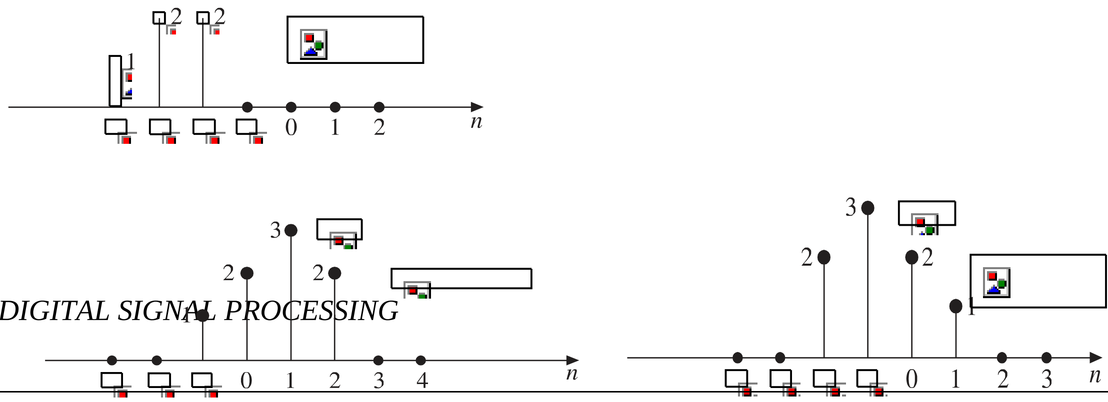


Figure 1.12 Time reversal operations.

and then multiplying these sequences element by element to obtain $u(n + 2)\,u(-n + 3)$ as shown in Figure 1.13(c).

$x(n) = 0$ for $n < -2$ and $n > 3$; $x(n) = 1$ for $-2 < n < 3$

(a) Given $x(n) = u(n + 4) - u(n - 2)$

The signal $u(n + 4) - u(n - 2)$ can be obtained by first plotting $u(n + 4)$ as shown in Figure 1.14(a), then plotting $u(n - 2)$ as shown in Figure 1.14(b), and then subtracting each element of $u(n - 2)$ from the corresponding element of $u(n + 4)$ to obtain the result shown in Figure 1.14(c).

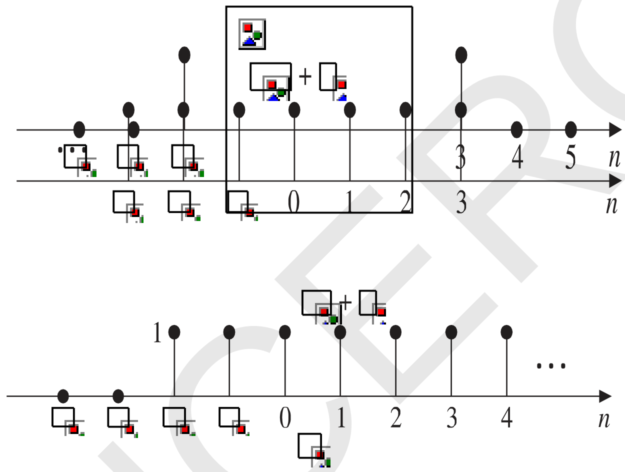
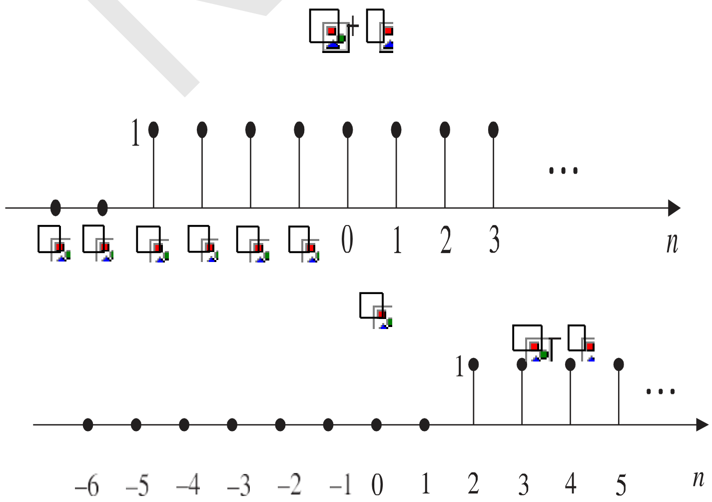


Figure 1.13 Plots of (a) u(n + 2) (b) u(–n + 3) (c) u(n + 2) u(–n + 3).



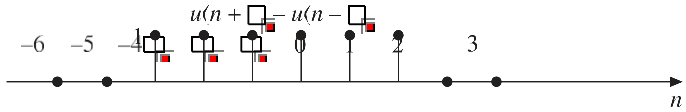


Figure 1.14 Plots of (a) u(n + 4) (b) u(n – 2) (c) u(n + 4) – u(n – 2).

1.4.3 Amplitude Scaling

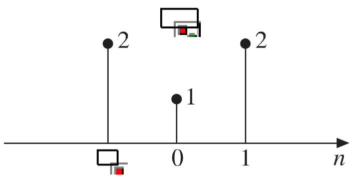
The amplitude scaling of a discrete-time signal can be represented by

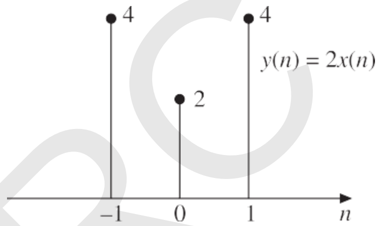
$$y(n) = ax(n)$$

where a is a constant.

The amplitude of $y(n)$ at any instant is equal to a times the amplitude of $x(n)$ at that instant. If $a > 1$, it is amplification and if $a < 1$, it is attenuation. Hence the amplitude is rescaled. Hence the name amplitude scaling.

Figure 1.15(a) shows a signal $x(n)$ and Figure 1.15(b) shows a scaled signal $y(n) = 2x(n)$.





1.4.1 Time Scaling

Time scaling may be time expansion or time compression. The time scaling of a discrete-time signal $x(n)$ can be accomplished by replacing n by an in it. Mathematically, it can be expressed as:

$$y(n) = x(an)$$

When $a > 1$, it is time compression and when $a < 1$, it is time expansion.

Let $x(n)$ be a sequence as shown in Figure 1.16(a). If $a = 2$, $y(n) = x(2n)$. Then

$$y(0) = x(0) = 1$$
$$y(-1) = x(-2) = 3$$
$$y(-2) = x(-4) = 0$$
$$y(1) = x(2) = 3$$
$$y(2) = x(4) = 0$$

and so on.

So to plot $x(2n)$ we have to skip odd numbered samples in $x(n)$.

We can plot the time scaled signal $y(n) = x(2n)$ as shown in Figure 1.16(b). Here the signal is compressed by 2.

If $a = (1/2)$, $y(n) = x(n/2)$, then

$$y(0) = x(0) = 1$$
$$y(2) = x(1) = 2$$
$$y(4) = x(2) = 3$$
$$y(6) = x(3) = 4$$
$$y(8) = x(4) = 0$$
$$y(-2) = x(-1) = 2$$
$$y(-4) = x(-2) = 3$$
$$y(-6) = x(-3) = 4$$

$$y(-8) = x(-4) = 0$$

We can plot $y(n) = x(n/2)$ as shown in Figure 1.16(c). Here the signal is expanded by 2. All Odd components in $x(n/2)$ are zero because $x(n)$ does not have any value in between the samplinginstants.

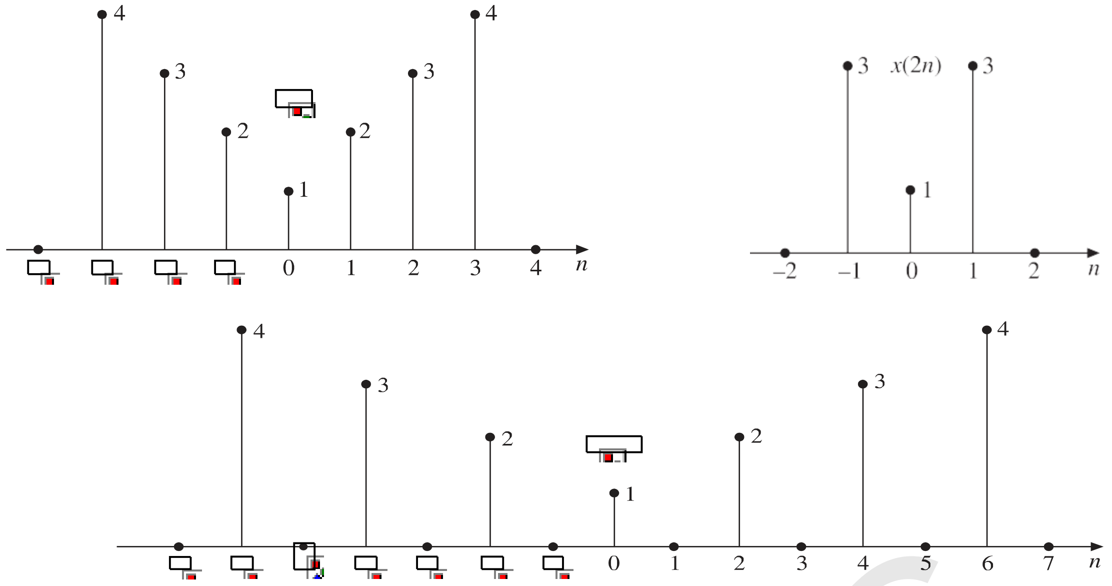


Figure 1.16 Discrete-time scaling (a) Plot of $x(n)$ (b) Plot of $x(2n)$ (c) Plot of $x(n/2)$

Time scaling is very useful when data is to be fed at some rate and is to be taken out at a differentrate.

1.45 Signal Addition

In discrete-time domain, the sum of two signals $x_1(n)$ and $x_2(n)$ can be obtained by adding the corresponding sample values and the subtraction of $x2(n)$ from $x1(n)$ can be obtained by subtracting each sample of $x2(n)$ from the corresponding sample of $x1(n)$ as illustrated below.

If $x_1(n) = \{1, 2, 3, 1, 5\}$ and $x_2(n) = \{2, 3, 4, 1, –2\}$

Then $x_1(n) + x_2(n) = \{1 + 2, 2 + 3, 3 + 4, 1 + 1, 5 – 2\} = \{3, 5, 7, 2, 3\}$

and $x_1(n) – x_2(n) = \{1 – 2, 2 – 3, 3 – 4, 1 – 1, 5 + 2\} = \{–1, –1, –1, 0, 7\}$

1.4.6 Signal multiplication

The multiplication of two discrete-time sequences can be performed by multiplying their values atthe sampling instants as shown below.

If $x1(n) = \{1, –3, 2, 4, 1.5\}$ and $x2(n) = \{2, –1, 3, 1.5, 2\}$

Then $x_1(n)\, x_2(n) = \{1 \times 2, -3 \times -1, 2 \times 3, 4 \times 1.5, 1.5 \times 2\}$

$= \{2, 3, 6, 6, 3\}$

EXAMPLE 1.3 Express the signals shown in Figure 1.17 as the sum of singular functions.

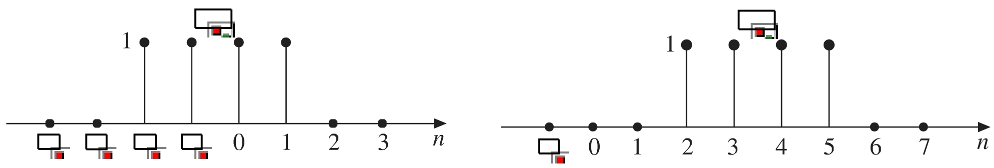


Figure 1.17 Waveforms for Example 1.3

Solution:

(a) The given signal shown in Figure 1.17(a) is:

x(n) = δ (n + 2) + δ (n + 1) + δ(n) + δ(n -
1)

0 $for\ n \leq -3$

$$x(n) = \begin{cases} 1 & for -2 \le n \le 1 \\ 0 & for\ n \ge 2 \end{cases}$$

∴ x(n) = u(n+2) – u(n-2)

The signal shown in Figure 1.17(b) is:

$$x(n) = \delta(n-2) + \delta(n-3) + \delta(n-4) + \delta(n-5)$$

$$x(n) = \begin{cases} 0 & for\ n \le 1 \\ 1 & for\ 2 \le n \le 5 \\ 0 & for\ n \ge 6 \end{cases}$$

∴ x(n) = u(n – 2) – u(n – 6)

1.4 CLASSIFICATION OF DISCRETE-TIME SIGNALS

The signals can be classified based on their nature and characteristics in the time domain. They are broadly classified as: (i) continuous-time signals and (ii) discrete-time signals.

The signals that are defined for every instant of time are known as continuous-time signals. The continuous-time signals are also called analog signals. They are denoted by *x* (*t*). They are continuous in amplitude as well as in time. Most of the signals available are continuous-time signals.

The signals that are defined only at discrete instants of time are known as discrete-time signals. The discrete-time signals are continuous in amplitude, but discrete in time. For discrete- time signals, the amplitude between two time instants is just not defined. For discrete-time signals, the independent variable is time *n*. Since they are defined only at discrete instants of time, they are denoted by a sequence *x* (*nT*) or simply by *x*(*n*) where *n* is an integer.

Figure 1.18 shows the graphical representation of discrete-time signals. The discrete-time signals may be inherently discrete or may be discrete versions of the continuous-time signals.

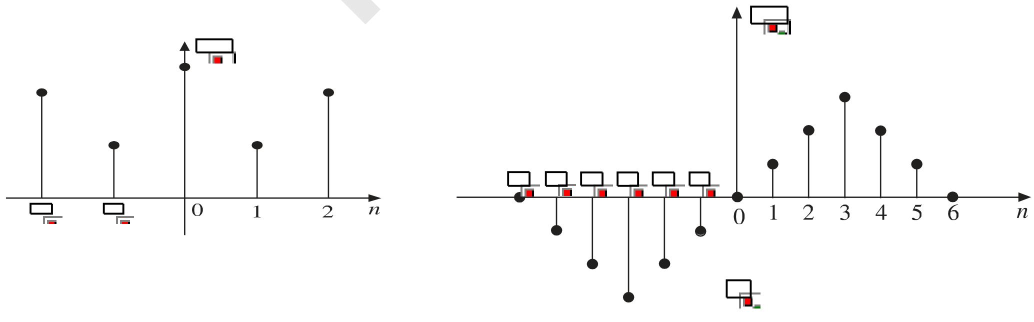


Figure 1.18 Discrete-time signals

Both continuous-time and discrete-time signals are further classified as follows:

1. Deterministic and random signals
2. Periodic and non-periodic signals
3. Energy and power signals
4. Causal and non-causal signals

5. Even and odd signals

1.5.1 Deterministic and Random Signals

A signal exhibiting no uncertainty of its magnitude and phase at any given instant of time is called deterministic signal. A deterministic signal can be completely represented by mathematical equationat any time and its nature and amplitude at any time can be predicted.

Examples: Sinusoidal sequence $x(n) = \cos n$, Exponential sequence $x(n) = e^{j n}$, ramp sequence $x(n) = n$.

A signal characterized by uncertainty about its occurrence is called a non-deterministic or random signal. A random signal cannot be represented by any mathematical equation. The behavior of such a signal is probabilistic in nature and can be analyzed only stochastically. The pattern of such a signal is quite irregular. Its amplitude and phase at any time instant cannot be predicted in advance. A typical example of a non-deterministic signal is thermal noise.

1.5.2 Periodic and Non-periodic Sequences

A signal which has a definite pattern and repeats itself at regular intervals of time is called a periodic signal, and a signal which does not repeat at regular intervals of time is called a non-periodic or aperiodic signal.

A discrete-time signal $x(n)$ is said to be periodic if it satisfies the condition $x(n) = x(n + N)$ for all integers n.

The smallest value of N which satisfies the above condition is known as fundamental period.

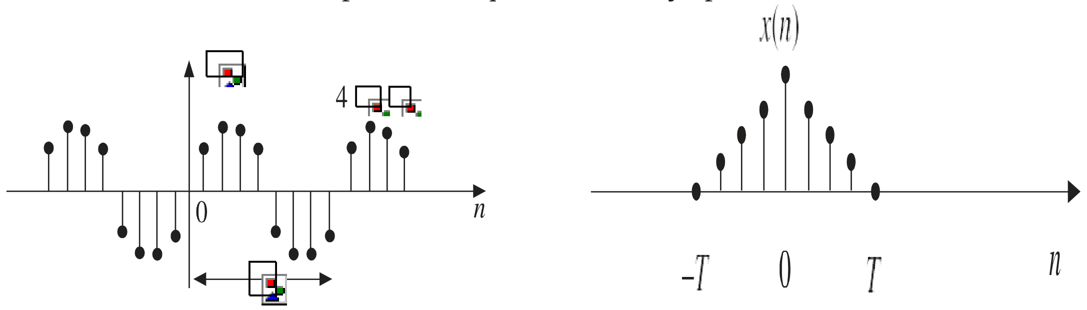
If the above condition is not satisfied even for one value of n, then the discrete-time signal is aperiodic. Sometimes aperiodic signals are said to have a period equal to infinity.

The angular frequency is given by

$$\omega = \frac{2\pi}{N}$$

Fundamental period $N = \frac{2\pi}{\omega}$

The sum of two discrete-time periodic sequence is always periodic.



some examples of discrete-time periodic/non-periodic signals are shown in Figure 1.19.
Figure 1.19 Example of discrete-time: (a) Periodic and (b) Non-periodic signals

EXAMPLE 1.4 Show that the complex exponential sequence $x(n) = e^{j\omega_0 n}$ is periodic only if $\omega_0/2\pi$ is a rational number.

Solution: Given $x(n) = e^{j\omega_0 n}$

X (n) will be periodic if $x(n + N) = x(n)$

i.e. $e^{j[\omega_0(n+N_0)]} = e^{j\omega_0 n}$

i.e. $e^{j\omega_0 N}\, e^{j\omega_0 n} = e^{j\omega_0 n}$

This is possible only if $e^{j\omega_0 N} = 1$

This is true only if $\omega_0 N = 2\pi k$

Where k is an integer $\frac{\omega_0}{2\pi} = \frac{k}{N}$

1.5.3 *Energy Signals And Power Signals*

Signals may also be classified as energy signals and power signals. However there are somesignals which can neither be classified as energy signals nor power signals. The total energy E of a discrete-time signal $x(n)$ is defined as:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

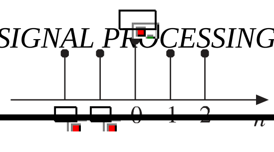
$$P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

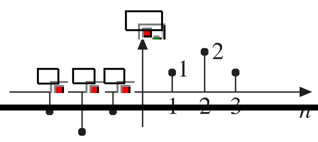
or $P = \frac{1}{N}\sum_{n=0}^{n-1} |x(n)|^2$ for a digital signal with $x(n) = 0$ for $n < 0$.

and the average power P of a discrete-time signal $x(n)$ is defined as:

A signal is said to be an energy signal if and only if its total energy E over the interval $(-\infty, \infty)$ is finite (i.e., $0 < E < \infty$). For an energy signal, average power $P = 0$. Non-periodic signals which are defined over a finite time (also called time limited signals) are the examples of energy signals. Since the energy of a periodic signal is always either zero or infinite, any periodic signal cannot be an energy signal.

A signal is said to be a power signal, if its average power P is finite (i.e., $0 < P < \infty$).





For a power signal, total energy $E = \infty$. Periodic signals are the examples of power signals. Every bounded and periodic signal is a power signal. But it is true that a power signal is not necessarily a bounded and periodic signal.

Both energy and power signals are mutually exclusive, i.e. no signal can be both energy signal and power signal.

The signals that do not satisfy the above properties are neither energy signals nor power signals. For example, $x(n) = u(n)$, $x(n) = nu(n)$, $x(n) = n^2u(n)$.

These are signals for which neither P nor E are finite. If the signals contain infinite energy and zero power or infinite energy and infinite power, they are neither energy nor power signals.

If the signal amplitude becomes zero as $|n| \to \infty$, it is an energy signal, and if the signal amplitude does not become zero as $|n| \to \infty$, it is a power signal.

Causal and Non-causal Signals

A discrete-time signal $x(n)$ is said to be causal if $x(n) = 0$ for $n < 0$, otherwise the signal is non- causal. A discrete-time signal $x(n)$ is said to be anti-causal if $x(n) = 0$ for $n > 0$.

A causal signal does not exist for negative time and an anti-causal signal does not exist for positive time. A signal which exists in positive as well as negative time is called a non-casual signal.

$u(n)$ is a causal signal and $u(-n)$ an anti-causal signal, whereas $x(n) = 1$ for $-2 \le n \le 3$ is a non-causal signal.

Even and Odd Signals

Any signal $x(n)$ can be expressed as sum of even and odd components. That is

$$x(n) = x_e(n) + x_o(n)$$

where $x_e(n)$ is even components and $x_o(n)$ is odd components of the signal.

Even (syMMetric) signal

A discrete-time signal $x(n)$ is said to be an even (symmetric) signal if it satisfies the condition:

$$x(n) = x(-n) \text{ for all } n$$

Even signals are symmetrical about the vertical axis or time origin. Hence they are also called symmetric signals: cosine sequence is an example of an even signal. Some even signals are shown in Figure 1.20(a). An even signal is identical to its reflection about the origin. For an even signal $x0(n) = 0$.

Odd (anti-syMMetric) signal

A discrete-time signal $x(n)$ is said to be an odd (anti-symmetric) signal if it satisfies the condition:

$$x(-n) = -x(n) \text{ for all } n$$

Odd signals are anti-symmetrical about the vertical axis. Hence they are called anti-symmetric signals. Sinusoidal sequence is an example of an odd signal. For an odd signal $xe(n) = 0$. Some odd signals are shown in Figure 1.20(b).

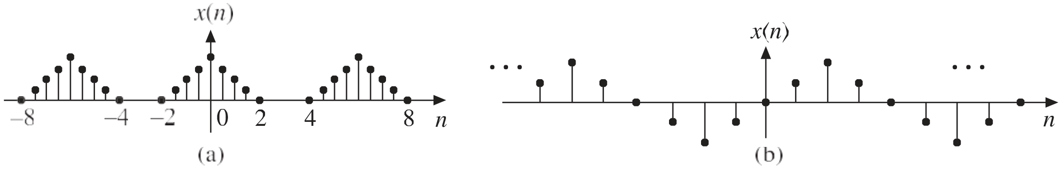


Figure 1.20 (a) Even sequences (b) Odd sequences.

Thus, the product of two even signals or of two odd signals is an even signal, and theproduct of even and odd signals is an odd signal.

Every signal need not be either purely even signal or purely odd signal, but everysignal can be decomposed into sum of even and odd parts.

CLASSIFICATION OF DISCRETE-TIME SYSTEMS

A system is defined as an entity that acts on an input signal and transforms it into an output signal. A system may also be defined as a set of elements or functional blocks which are connected together and produces an output in response to an input signal. The response or output of the system depends on the transfer function of the system. It is a cause and effect relation between two or more signals.

As signals, systems are also broadly classified into continuous-time and discrete-time systems. A continuous-time system is one which transforms continuous-time input signals into continuous-time output signals, whereas a discrete-time system is one which transforms discrete-time input signals into discrete-time output signals.

For example microprocessors, semiconductor memories, shift registers, etc. are discrete- time systems.

A discrete-time system is represented by a block diagram as shown in Figure 1.22. An arrow entering the box is the input signal (also called excitation, source or driving function) and an arrow leaving the box is an output signal (also called response). Generally, the inputis denoted by $x(n)$ and the output is denoted by $y(n)$.

The relation between the input $x(n)$ and the output $y(n)$ of a system has the form:

$$y(n) = \text{Operation on } x(n)$$

Mathematically,

$$y(n) = T[x(n)]$$

which represents that $x(n)$ is transformed to $y(n)$. In other words, $y(n)$ is the transformedversion of $x(n)$.

Figure 1.22 Block diagram of discrete-time system.

Both continuous-time and discrete-time systems are further classified as follows:

1. Static (memoryless) and dynamic (memory) systems
2. Causal and non-causal systems
3. Linear and non-linear systems
4. Time-invariant and time varying systems
5. Stable and unstable systems.
6. Invertible and non-invertible systems
7. FIR and IIR systems

Static and Dynamic Systems

A system is said to be static or memoryless if the response is due to present input alone, i.e., for a static or memoryless system, the output at any instant n depends only on the input applied at that instant n but not on the past or future values of input or past values of output.

For example, the systems defined below are static or memoryless systems.

$$y(n) = x(n)$$

$$y(n) = 2x^2(n)$$

In contrast, a system is said to be dynamic or memory system if the response depends upon past or future inputs or past outputs. A summer or accumulator, a delay element is a discrete- time system with memory.

For example, the systems defined below are dynamic or memory systems.

$$y(n) = x(2n)$$

$$y(n) = x(n) + x(n-2)$$

$$y(n) + 4y(n-1) + 4y(n-2) = x(n)$$

Any discrete-time system described by a difference equation is a dynamic system.

A purely resistive electrical circuit is a static system, whereas an electric circuit having inductors and/or capacitors is a dynamic system.

A discrete-time LTI system is memoryless (static) if its impulse response $h(n)$ is zero for n s 0. If the impulse response is not identically zero for n s 0, then the system is called dynamic system or system with memory.

EXAMPLE 1.12 Find whether the following systems are dynamic or not: (a) $y(n) = x(n + 2)$ (b) $y(n) = x^2(n)$
(c) $y(n) = x(n-2) + x(n)$

Solution:

(a) Given $y(n) = x(n + 2)$

The output depends on the future value of input. Therefore, the system is dynamic.

(b) Given $y(n) = x^2(n)$

The output depends on the present value of input alone. Therefore, the system is static.

(c) Given $y(n) = x(n-2) + x(n)$

The system is described by a difference equation. Therefore, the system is dynamic.

Causal and Non-causal Systems

A system is said to be causal (or non-anticipative) if the output of the system at any instant n depends only on the present and past values of the input but not on future inputs, i.e., for a causal system, the impulse response or output does not begin before the input function is applied, i.e., a causal system is non anticipatory.

Causal systems are real time systems. They are physically realizable.

The impulse response of a causal system is zero for $n < 0$, since (n) exists only at $n = 0$,

i.e. $$h(n) = 0 \text{ for } n < 0$$

The examples for causal systems are:

$$y(n) = nx(n)$$

$$y(n) = x(n-2) + x(n-1) + x(n)$$

A system is said to be non-causal (anticipative) if the output of the system at any instant n depends on future inputs. They are anticipatory systems. They produce an output even before the input is given. They do not exist in real time. They are not physically realizable.

A delay element is a causal system, whereas an image processing system is a non-causal system.

The examples for non-causal systems are:

$$y(n) = x(n) + x(2n)$$

$$y(n) = x^2(n) + 2x(n + 2)$$

Superposition property means a system which produces an output $y_1(n)$ for an input $x_1(n)$ and an output $y_2(n)$ for an input $x_2(n)$ must produce an output $y_1(n) + y_2(n)$ for an input $x_1(n) + x_2(n)$.

Combining them we can say that a system is linear if an arbitrary input $x_1(n)$ produces an output $y_1(n)$ and an arbitrary input $x_2(n)$ produces an output $y_2(n)$, then the weighted sum of inputs $ax_1(n) + bx_2(n)$ where a and b are constants produces an output $ay_1(n) + by_2(n)$ which is the sum of weighted outputs.

EXAMPLE 1.13 Check whether the following systems are causal or not: (a) $y(n)\ x(n)\ x(n\ 2)$ (b) $y(n) = x(2n)$
(c) $y(n) = \sin[x(n)]$ (d) $y(n) = x(-n)$

Solution:

(a) Given

$y(n)\ x(n)\ x(n\ 2)$

For $n = -2$ $y(\ 2)\ x(\ 2)\ x(\ 4)$

For $n = 0$ $y(0)\ x(0)\ x(\ 2)$

For $n = 2$ $y(2)\ x(2)\ x(0)$

For all values of n, the output depends only on the present and past inputs. Therefore, the system is causal.

(a) Given $y(n)\ \ x(2n)$

For $n = -2$ $y(\ 2)\ \ x(\ 4)$

For $n = 0$ $y(0)\ \ x(0)$

For $n = 2$ $y(2)\ \ x(4)$

For positive values of n, the output depends on the future values of input. Therefore, the system is non-causal.

(a) Given $y(n)\ \sin[x(n)]$

For $n = -2$ $y(\ 2)\ \sin[x(\ 2)]$

For $n = 0$

For $n = 2$

$y(0)$ sin $[x(0)]$ $y(2)$ sin $[x(2)]$

For all values of n, the output depends only on the present value of input. Therefore,the system is causal.

(d) Given $y(n) = x(-n)$

For $n = -2$

$y(\ 2)\ x(2)$

For $n = 0$ $y(0)\ x(0)$

For $n = 2$ $y(2)\ x(\ 2)$

For negative values of n, the output depends on the future values of input. Therefore, the system is non-causal.

Linear and Non-linear Systems

A system which obeys the principle of superposition and principle of homogeneity is called a linear system and a system which does not obey the principle of superposition and homogeneity is called a non-linear system.

Homogeneity property means a system which produces an output $y(n)$ for an input $x(n)$ must produce an output $ay(n)$ for an input $ax(n)$.

$$T(ax_1(n) + bx_2(n)] = aT[x_1(n)] + bT[x_2(n)]$$

Simply we can say that a system is linear if the output due to weighted sum of inputs is equal to the weighted sum of outputs.

In general, if the describing equation contains square or higher order terms of input and/or output and/or product of input/output and its difference or a constant, the system will definitely be non-linear.

Shift-invariant and Shift-varying Systems

Time-invariance is the property of a system which makes the behaviour of the system independent of time. This means that the behaviour of the system does not depend on the time at which the input is applied. For discrete-time systems, the time invariance property is called shift invariance.

A system is said to be shift-invariant if its input/output characteristics do not change with time, i.e., if a time shift in the input results in a corresponding time shift in the output as shown in Figure 1.23, i.e.

If $T[x(n)] = y(n)$

Then $T[x(n - k)] = y(n - k)$

A system not satisfying the above requirements is called a time-varying system (or shift-varying system). A time-invariant system is also called a fixed system.

The time-invariance property of the given discrete-time system can be tested asfollows:

Let $x(n)$ be the input and let $x(n - k)$ be the input delayed by k units.

$y(n) = T[x(n)]$ be the output for the input $x(n)$.

Stable and Unstable Systems

A bounded signal is a signal whose magnitude is always a finite value, i.e. $|x(n)| \leq M$, where M is a positive real finite number. For example a sinewave is a bounded signal. A system is said to be bounded-input, bounded-output (BIBO) stable, if and only if every bounded input produces a bounded output. The output of such a system does not diverge or does not grow unreasonably large.

Let the input signal $x(n)$ be bounded (finite), i.e.,

$$|x(n)| \leq M_x \quad \text{for all } n$$

where M_x is a positive real number. If

$$|y(n)| \leq M_y \leq \infty$$

i.e. if the output $y(n)$ is also bounded, then the system is BIBO stable. Otherwise, the system is unstable. That is, we say that a system is unstable even if one bounded input produces an unbounded output.

It is very important to know about the stability of the system. Stability indicates the usefulness of the system. The stability can be found from the impulse response of the system which is nothing but the output of the system for a unit impulse input. If the impulse response is absolutely summable for a discrete-time system, then the system is stable.

BlBO stability criterion

The necessary and sufficient condition for a discrete-time system to be BIBO stable is givenby the expression:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

where $h(n)$ is the impulse response of the system. This is called BIBO stability criterion.

Proof: Consider a linear time-invariant system with $x(n)$ as input and $y(n)$ as output. Theinput and output of the system are related by the convolution integral.

SOLUTION OF DIFFERENCE EQUATIONS USING Z-TRANSFORMS.

To solve the difference equation, first it is converted into algebraic equation by taking its Z-transform. The solution is obtained in *z*-domain and the time domain solution is obtained by taking its inverse Z-transform. The system response has two components. The source free response and the forced response. The response of the system due to input alone when the initial conditions are neglected is called the forced response of the system. It is also called the steady state response of the system. It represents the component of the response due to the driving force. The response of the system due to initial conditions alone when the input is neglected is called the free or natural response of the system. It is also called the transient response of the system. It represents the component of the response when the driving function is made zero. The response due to input and initial conditions considered simultaneously is called the total response of the system. For a stable system, the source free component always decays with time. In fact a stable system is one whose source free component decays with time. For this reason the source free component is also designated as the transient component and the component due to source is called the steady state component. When input is a unit impulse input, the response is called the impulse response of the system and when the input is a unit step input, the response is called the step response of the system.

EXAMPLE 1 A linear shift invariant system is described by the difference equation

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + x(n-1)$$

with $y(-1) = 0$ and $y(-2) = -1$.

Find (a) the natural response of the system (b) the forced response of the system for a step input and (c) the frequency response of the system.

Solution:

(a) The natural response is the response due to initial conditions only. So make $x(n) = 0$. Then the difference equation becomes

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = 0$$

Taking Z-transform on both sides, we have

$$Y(z) - \frac{3}{4}[z^{-1}Y(z) + y(-1)] + \frac{1}{8}[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)] = 0$$

i.e. $$Y(z)\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) - \frac{1}{8} = 0$$

$$\therefore \quad Y(z) = \frac{1/8}{1-(3/4)z^{-1} + (1/8)z^{-2}} = \frac{1/8z^2}{z^2 - (3/4)z + (1/8)} = \frac{1/8z^2}{[z-(1/2)][z-(1/4)]}$$

The partial fraction expansion of $Y(z)/z$ gives

$$\frac{Y(z)}{z} = \frac{(1/8)z}{[z-(1/2)][z-(1/4)]} = \frac{A}{z-(1/2)} + \frac{B}{z-(1/4)} = \frac{1/4}{z-(1/2)} - \frac{1/8}{z-(1/4)}$$

$$Y(z) = \frac{1}{4}\frac{z}{z-(1/2)} - \frac{1}{8}\frac{z}{z-(1/4)}$$

Taking inverse Z-transform on both sides, we get the natural response as:

i.e. $$Y(z)\left(1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\right)=\frac{z+1}{z-1}$$

$$\therefore\quad Y(z)=\frac{z+1}{(z-1)\,[1-(3/4)z^{-1}+(1/8)z^{-2})}=\frac{z^2(z+1)}{(z-1)[z^2-(3/4)z+(1/8)]}$$

$$=\frac{z^2(z+1)}{(z-1)\,[z-(1/2)]\,[z-(1/4)]}$$

Taking partial fractions of $Y(z)/z$, we have

$$\therefore\quad \frac{Y(z)}{z}=\frac{z(z+1)}{(z-1)\,[z-(1/2)]\,[z-(1/4)]}=\frac{A}{z-1}+\frac{B}{z-(1/2)}+\frac{C}{z-(1/4)}$$

$$=\frac{16/3}{z-1}-\frac{6}{z-(1/2)}+\frac{5/3}{z-(1/4)}$$

or $$Y(z)=\frac{16}{3}\left(\frac{z}{z-1}\right)-6\left[\frac{z}{z-(1/2)}\right]+\frac{5}{3}\left[\frac{z}{z-(1/4)}\right]$$

Taking the inverse Z-transform on both sides, we have the forced response for astep input.

© The frequency response of the system $H(\ )$ is obtained by putting $z = e^{jw}$ in $H(z)$.

$$H(z)=\frac{Y(z)}{X(z)}=\frac{z(z+1)}{z^2-(3/4)\,z+(1/8)}$$

$$H(\omega)=\frac{e^{j\omega}\,(e^{j\omega}+1)}{(e^{j\omega})^2-(3/4)\,e^{j\omega}+(1/8)}$$

EXAMPLE 2 (a) Determine the free response of the system described by the difference equation

(a) Determine the forced response for an input

Solution:

(a) The free response, also called the natural response or transient response is the response due to initial conditions only [i.e. make $x(n) = 0$]. So, the difference equation is:

$$y(n)-\frac{5}{6}y(n-1)+\frac{1}{6}y(n-2)=0$$

Taking Z-transform on both sides, we get

$$Y(z)-\frac{5}{6}[z^{-1}Y(z)+y(-1)]+\frac{1}{6}[z^{-2}Y(z)+z^{-1}y(-1)+y(-2)]=0$$

$$Y(z)\left(1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}\right)-\frac{5}{6}+\frac{1}{6}z^{-1}=0$$

$$\therefore \quad Y(z)=\frac{(5/6)-(1/6)z^{-1}}{1-(5/6)z^{-1}+(1/6)z^{-2}}=\frac{5/6[z-(1/5)]z}{z^2-(5/6)z+(1/6)}=\frac{(5/6)z[z-(1/5)]}{[z-(1/2)][z-(1/3)]}$$

Taking partial fractions of $Y(z)/z$, we have

$$\frac{Y(z)}{z}=\frac{5/6[z-(1/5)]}{[z-(1/2)][z-(1/3)]}=\frac{A}{z-(1/2)}+\frac{B}{z-(1/3)}=\frac{3/2}{z-(1/2)}-\frac{2/3}{z-(1/3)}$$

$$\therefore \quad Y(z)=\frac{3}{2}\frac{z}{z-(1/2)}-\frac{2}{3}\frac{z}{z-(1/3)}$$

Taking inverse Z-transform on both sides, we get the free response of the system as:

(a) To determine the forced response, i.e. the steady state response, the initial conditionsare to be neglected.

The given difference equation is:

$$y(n)-\frac{5}{6}y(n-1)+\frac{1}{6}y(n-2)=x(n)=\left(\frac{1}{4}\right)^n u(n)$$

Taking Z-transform on both sides and neglecting the initial conditions, we have

$$Y(z)-\frac{5}{6}z^{-1}Y(z)+\frac{1}{6}z^{-2}Y(z)=\frac{z}{z-(1/4)}$$

i.e.,

$$Y(z)\left(1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}\right)=\frac{z}{z-(1/4)}$$

$$\therefore \quad Y(z)=\frac{z}{z-(1/4)}\frac{1}{1-(5/6)z^{-1}+(1/6)z^{-2}}=\frac{z^3}{[z-(1/4)][z-(1/2)][z-(1/3)]}$$

Partial fraction expansion of $Y(z)/z$ gives

$$\frac{Y(z)}{z}=\frac{z^2}{[z-(1/4)][z-(1/3)][z-(1/2)]}=\frac{A}{z-(1/4)}+\frac{B}{z-(1/3)}+\frac{C}{z-(1/2)}$$

$$=\frac{3}{z-(1/4)}-\frac{8}{z-(1/3)}+\frac{6}{z-(1/2)}$$

Multiplying both sides by z, we get

$$Y(z) = 3\frac{z}{z-(1/4)} - 8\frac{z}{z-(1/3)} + 6\frac{z}{z-(1/2)}$$

Taking inverse Z-transform on both sides, the forced response of the system is:

$$y(n) = 3\left(\frac{1}{4}\right)^n u(n) - 8\left(\frac{1}{3}\right)^n u(n) + 6\left(\frac{1}{2}\right)^n u(n)$$

EXAMPLE 3 Find the impulse and step response of the system $y(n) = 2x(n) - 3x(n-1) + x(n-2) - 4x(n-3)$

Solution: For impulse response, $x(n) = \delta(n)$ The impulse response of the system is:

$$y(n) = 2\delta(n) - 3\delta(n-1) + \delta(n-2) - 4\delta(n-3)$$

For step response, $x(n) = u(n)$

The step response of the system

is:

$$y(n) = 2u(n) - 3u(n\ 1) + u(n\ 2) - 4u(n\ 3)$$

EXAMPLE 4 Solve the following difference equation

$$y(n) + 2y(n-1) = x(n)$$

with $x(n) = (1/3)^n u(n)$ and the initial condition $y(-1)$

= 1.

Solution: The solution of the difference equation considering the initial condition and inputsimultaneously gives the total response of the system. The given difference equation is:

$$y(n) + 2y(n-1) = x(n) = \left(\frac{1}{3}\right)^n u(n) \text{ with } y(-1) = 1$$

Taking Z-transform on both sides, we get

$$Y(z) + 2[z^{-1}Y(z) + y(-1)] = X(z) = \frac{1}{1-(1/3)z^{-1}}$$

Substituting the initial conditions, we have

$$Y(z)(1 + 2z^{-1}) = -2(1) + \frac{1}{1-(1/3)z^{-1}}$$

$$\therefore \quad Y(z) = \frac{-2}{1+2z^{-1}} + \frac{1}{[1-(1/3)z^{-1}][1+2z^{-1}]}$$

$$= \frac{-2z}{z+2} + \frac{z^2}{[z-(1/3)](z+2)}$$

Let $\quad Y_1(z) = \frac{z^2}{[z-(1/3)](z+2)}$

Taking partial fractions of $Y_1(z)/z$, we have

$$\frac{Y_1(z)}{z} = \frac{z}{[z-(1/3)]\,(z+2)} = \frac{A}{z-(1/3)} + \frac{B}{z+2} = \frac{1/7}{z-(1/3)} + \frac{6/7}{z+2}$$

Multiplying both sides by z, we have

Taking inverse Z-transform on both sides, the solution of the difference equation is:

EXAMPLE 5 Solve the following difference equation using unilateral Z-transform. with

initialconditions

Solution: The solution of the difference equation gives the total response of the system(i.e., the sum of the natural (free) response and the forced response)

The given difference equation is:

$$y(n) - \frac{7}{12}y(n-1) + \frac{1}{12}y(n-2) = x(n) = \left(\frac{1}{5}\right)^n u(n)$$

with initial conditions $y(-1) = 2$ and $y(-2) = 4$. Taking Z-transform on both sides, we have

$$Y(z) - \frac{7}{12}[z^{-1}Y(z) + y(-1)] + \frac{1}{12}[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)] = \frac{1}{1-(1/5)z^{-1}}$$

i.e. $$Y(z)\left(1 - \frac{7}{12}z^{-1} + \frac{1}{12}z^{-2}\right) = \frac{7}{12}(2) - \frac{1}{12}(2z^{-1}) - \frac{1}{12}(4) + \frac{1}{1-(1/5)z^{-1}}$$

i.e. $$Y(z)\left(1 - \frac{7}{12}z^{-1} + \frac{1}{12}z^{-2}\right) = \frac{5}{6}\left(1 - \frac{1}{5}z^{-1}\right) + \frac{1}{1-(1/5)z^{-1}}$$

$$\therefore\ Y(z) = \frac{(5/6)\,[1-(1/5)\,z^{-1}]}{[1-(7/12)z^{-1} + (1/12)z^{-2}]} + \frac{1}{[1-(1/5)z^{-1}]\,[1-(7/12)\,z^{-1} + (1/12)z^{-2}]}$$

$$= \frac{(5/6)\,[z-(1/5)]\,z}{[z-(1/4)]\,[z-(1/3)]} + \frac{z^3}{[z-(1/5)]\,[z-(1/4)]\,[z-(1/3)]}$$

$$= \frac{z\,[(11/6)\,z^2 - (1/3)z + (1/30)]}{[z-(1/5)]\,[z-(1/4)][z-(1/3)]}$$

Taking partial fractions of $Y(z)/z$, we have

$$\frac{Y(z)}{z}=\frac{A}{z-(1/5)}+\frac{B}{z-(1/4)}+\frac{C}{z-(1/3)}=\frac{6}{5}\frac{1}{z-(1/5)}+\frac{1}{8}\frac{1}{z-(1/4)}+\frac{100}{27}\frac{1}{z-(1/3)}$$

Multiplying both sides by z, we have

$$Y(z)=\frac{6}{5}\frac{z}{z-(1/5)}+\frac{1}{8}\frac{z}{z-(1/4)}+\frac{102}{27}\frac{z}{z-(1/3)}$$

Taking inverse Z-transform on both sides, the solution of the difference equation is:

$$y(n)=\frac{6}{5}\left(\frac{1}{5}\right)^n u(n)+\frac{1}{8}\left(\frac{1}{4}\right)^n u(n)+\frac{102}{27}\left(\frac{1}{3}\right)^n u(n)$$

EXAMPLE 6 Using Z-transform determine the response of the LTI system described by $y(n) \square 2r \cos y(n \square 1) + r^2 y(n \square 2) = x(n)$ to an excitation $x(n) = a^n u(n)$.

Solution: Taking Z-transform on both sides of the difference equation, we have

$$Y(z)-2r\cos\theta\,[z^{-1}Y(z)+y(-1)]+r^2[z^{-2}Y(z)+z^{-1}y(-1)+y(-2)]=X(z)$$

i.e. $$Y(z)\,[1-2r\cos\theta\, z^{-1}+r^2z^{-2}]=\frac{z}{z-a}$$

$$\therefore \quad Y(z)=\frac{z^3}{(z-a)\,(z-re^{j\theta})\,(z-re^{-j\theta})}$$

$$=\frac{a^2}{a^2-2ar\cos\theta+r^2}\frac{z}{z-a}+\frac{r^2e^{j2\theta}}{(re^{j\theta}-a)(re^{j\theta}-re^{-j\theta})}\frac{z}{z-re^{j\theta}}$$
$$+\frac{r^2e^{-j2\theta}}{(re^{-j\theta}-a)(re^{-j\theta}-re^{j\theta})}\frac{z}{z-re^{-j\theta}}$$

$$\therefore \quad y(n)=\frac{a^2}{a^2-2ar\cos\theta+r^2}a^n u(n)+\frac{r^{n+1}}{\sin\theta}\left[\frac{r\sin(n+1)\theta-a\sin(n+2)\theta}{a^2-2ar\cos\theta+r^2}\right]u(n)$$

EXAMPLE 7 Determine the step response of an LTI system whose impulse response $h(n)$ is given by $h(n) = a^{\square n}u(\square n); 0 < a < 1$.

Solution: The impulse response is

$$h(n)=a^{-n}u(-n);\,0<a<1$$

$$H(z)=\frac{1}{1-az}=-\frac{1}{a}\frac{1}{z-(1/a)}$$

$$x(n) = u(n) \quad \text{and} \quad H(z) = \frac{z}{z-1}$$

We have to find the step response

The step response of the system is given by

$$y(n) = x(n) * h(n)$$

$$Y(z) = X(z)\,H(z) = \left(-\frac{1}{a}\right)\frac{z}{z-1}\,\frac{1}{z-(1/a)} = \frac{1}{1-a}\left[\frac{z}{z-1} - \frac{z}{z-(1/a)}\right]$$

So the step response is

$$y(n) = \frac{1}{1-a}\left[u(n) - \left(\frac{1}{a}\right)^n u(n)\right]$$

EXAMPLE 8 The step response of an LTI system is

Solution: We have $s(n) = h(n) * u(n)$

$$\therefore \quad S(z) = H(z)U(z) = H(z)\frac{z}{z-1}$$

Given

$$s(n) = \left(\frac{1}{3}\right)^{n-2} u(n+2)$$

$$S(z) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{3}\right)^{n-2} u(n+2)\,z^{-n} = 3^2 \sum_{n=-2}^{\infty}\left(\frac{1}{3z}\right)^n$$

$$= 3^2\,\frac{\left(\frac{1}{3z}\right)^{-2}}{1-\frac{1}{3z}} = \frac{3^4 z^2}{1-\frac{1}{3}z^{-1}} = \frac{81z^3}{\left(z-\frac{1}{3}\right)}$$

The system function $H(z)$ is

$$H(z) = S(z)\frac{z-1}{z} = \frac{81z^3}{\left(z-\frac{1}{3}\right)}\frac{z-1}{z} = \frac{81z^2(z-1)}{\left(z-\frac{1}{3}\right)} = \frac{81z^3}{z-\frac{1}{3}} - \frac{81z^2}{z-\frac{1}{3}} = 81z^2\frac{z}{z-\frac{1}{3}} - 81z\frac{z}{z-\frac{1}{3}}$$

The impulse response of the system is

$$h(n) = 81\left(\frac{1}{3}\right)^{n+2} u(n+2) - 81\left(\frac{1}{3}\right)^{n+1} u(n+1) \quad = 9\left(\frac{1}{3}\right)^n u(n+2) - 27\left(\frac{1}{3}\right)^n u(n+1)$$

MODULE II

Discrete Fourier Transforms

INTRODUCTION :The DFT of a discrete-time signal $x(n)$ is a finite duration discrete frequencysequence. The DFT sequence is denoted by $X(k)$. The DFT is obtained by sampling one period of the Fourier transform $X(W)$ of the signal $x(n)$ at a finite number of frequency points. This sampling isconventionally performed at N equally spaced points in the period $0 \leq w \leq 2w$ or at $wk = 2\pi k/N$;
$0 \leq k \leq N - 1$. We can say that DFT is used for transforming discrete-time sequence $x(n)$ of finite lengthinto discrete frequency sequence $X(k)$ of finite length. The DFT is important for two reasons. First itallows us to determine the frequency content of a signal, that is to perform spectral analysis. Thesecond application of the DFT is to perform filtering operation in the frequency domain. Let $x(n)$ be adiscrete-time sequence with Fourier

$$X(k) = X(\omega)\big|_{\omega=(2\pi k/N)}; \quad \text{for } k = 0, 1, 2, \ldots, N-1$$

transform $X(W)$, then the DFT of $x(n)$ denoted by $X(k)$ is defined as

The DFT of $x(n)$ is a sequence consisting of N samples of $X(k)$. The DFT sequence starts at $k = 0$, corresponding to $w = 0$, but does not include $k = N$ corresponding to $w = 2\pi$ (since the sample at $w = 0$is same as the sample at $w = 2\pi$). Generally, the DFT is defined as

DFT The N-point DFT of a finite duration sequence $x(n)$ of length L, where $N \geq L$ is defined as:

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N} = \sum_{n=0}^{N-1} x(n)W_N^{nk}; \quad \text{for } k = 0, 1, 2, \ldots, N-1$$

IDFT The Inverse Discrete Fourier transform (IDFT) of the sequence $X(k)$ of length N is defined as:

$$\text{IDFT}\{X(k)\} = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi nk/N} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-nk}; \quad \text{for } n = 0, 1, 2, \ldots, N-1$$

where $W_N = e^{-j(2\pi/N)}$ is called the twiddle factor.

The N-point DFT pair $x(n)$ and $X(k)$ is denoted as:

$$x(n) \xleftrightarrow[N]{\text{DFT}} X(k)$$

EXAMPLE 2.1 (a) Find the 4-point DFT of $x(n) = \{1, –1, 2, –2\}$ directly.
(b) Find the IDFT of $X(k) = \{4, 2, 0, 4\}$ directly.

Solution:

(a) Given sequence is $x(n) = \{1, –1, 2, –2\}$. Here the DFT $X(k)$ to be found is N =4-pointand length of the sequence $L = 4$. So no padding of zeros is required. We know that the DFT
$\{x(n)\}$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n=0}^{N-1} x(n)\, e^{-j(2\pi/N)nk} = \sum_{n=0}^{3} x(n)\, e^{-j(\pi/2)nk}, \quad k = 0, 1, 2, 3$$

$$X(0) = \sum_{n=0}^{3} x(n)\, e^{0} = x(0) + x(1) + x(2) + x(3) = 1 - 1 + 2 - 2 = 0$$

$$X(1) = \sum_{n=0}^{3} x(n)\, e^{-j(\pi/2)n} = x(0) + x(1)\, e^{-(j\pi/2)} + x(2)\, e^{-j\pi} + x(3)\, e^{-j(3\pi/2)}$$

$$= 1 + (-1)(0 - j) + 2(-1 - j0) - 2(0 + j)$$

$$= -1 - j$$

$$X(2) = \sum_{n=0}^{3} x(n)\, e^{-j\pi n} = x(0) + x(1)\, e^{-j\pi} + x(2)\, e^{-j2\pi} + x(3)\, e^{-j3\pi}$$

$$= 1 - 1(-1 - j0) + 2(1 - j0) - 2(-1 - j0) = 6$$

$$X(3) = \sum_{n=0}^{3} x(n)\, e^{-j(3\pi/2)n} = x(0) + x(1)\, e^{-j(3\pi/2)} + x(2)\, e^{-j3\pi} + x(3)\, e^{-j(9\pi/2)}$$

$$= 1 - 1(0 + j) + 2(-1 - j0) - 2(0 - j) = -1 + j$$

$X(k) = \{0, -1 - j, 6, -1 + j\}$

(b) Given DFT is $X(k) = \{4, 2, 0, 4\}$. The IDFT of $X(k)$, i.e. $x(n)$ is given by

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk} = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j(2\pi/N)nk}$$

i.e.
$$x(n) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{j(\pi/2)nk}$$

$\therefore$
$$x(0) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{0} = \frac{1}{4}[X(0) + X(1) + X(2) + X(3)]$$

$$= \frac{1}{4}[4 + 2 + 0 + 4] = 2.5$$

$$x(1) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, e^{j(\pi/2)k} = \frac{1}{4}\left[X(0) + X(1)\, e^{j(\pi/2)} + X(2) e^{j\pi} + X(3) e^{j(3\pi/2)}\right]$$

$$= \frac{1}{4}[4 + 2(0 + j) + 0 + 4(0 - j)] = 1 - j0.5$$

$$x(2) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, e^{j\pi k} = \frac{1}{4}\left[X(0) + X(1)\, e^{j\pi} + X(2)\, e^{j2\pi} + X(3) e^{j3\pi}\right]$$

$$= \frac{1}{4}[4 + 2(-1 + j0) + 0 + 4(-1 + j0)] = -0.5$$

$$x(3) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, e^{j(3\pi/2)k} = \frac{1}{4}\left[X(0) + X(1)\, e^{j(3\pi/2)} + X(2)\, e^{j3\pi} + X(3) e^{j(9\pi/2)}\right]$$

$$= \frac{1}{4}[4 + 2(0 - j) + 0 + 4(0 + j)] = 1 + j0.5$$

$x_3(n) = \{2.5,\ 1 - j0.5,\ -0.5,\ 1 + j0.5\}$

EXAMPLE 2.2 (a) Find the 4-point DFT of $x(n) = \{1, -2, 3, 2\}$.
(b) Find the IDFT of $X(k) = \{1, 0, 1, 0\}$.

Solution:

(a) Given $x(n) = \{1, -2, 3, 2\}$.

Here $N = 4$, $L = 4$. The DFT of $x(n)$ is $X(k)$.

$$\therefore \quad X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n=0}^{3} x(n) e^{-j(2\pi/4)nk} = \sum_{n=0}^{3} x(n) e^{-j(\pi/2)nk}, \quad k = 0, 1, 2, 3$$

$$X(0) = \sum_{n=0}^{3} x(n)e^0 = x(0) + x(1) + x(2) + x(3) = 1 - 2 + 3 + 2 = 4$$

$$X(1) = \sum_{n=0}^{3} x(n)e^{-j(\pi/2)n} = x(0) + x(1)e^{-j(\pi/2)} + x(2)e^{-j\pi} + x(3)e^{-j(3\pi/2)}$$

$$= 1 - 2(0 - j) + 3(-1 - j0) + 2(0 + j) = -2 + j4$$

$$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n} = x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$

$$= 1 - 2(-1 - j0) + 3(1 - j0) + 2(-1 - j0) = 4$$

$$X(3) = \sum_{n=0}^{3} x(n)e^{-j(3\pi/2)n} = x(0) + x(1)e^{-j(3\pi/2)} + x(2)e^{-j3\pi} + x(3)e^{-j(9\pi/2)}$$

$$= 1 - 2(0 + j) + 3(-1 - j0) + 2(0 - j) = -2 - j4$$

$$\therefore \quad X(k) = \{4, -2 + j4, 4, -2 - j4\}$$

(b) Given $X(k) = \{1, 0, 1, 0\}$

Let the IDFT of $X(k)$ be $x(n)$.

$$\therefore \quad x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk} = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j(2\pi/N)nk}$$

$$x(0) = \frac{1}{4} \sum_{k=0}^{3} X(k)e^0 = \frac{1}{4}[X(0) + X(1) + X(2) + X(3)] = \frac{1}{4}[1 + 0 + 1 + 0] = 0.5$$

$$x(1) = \frac{1}{4} \sum_{k=0}^{3} X(k)e^{j(\pi/2)k} = \frac{1}{4}\left[X(0) + X(1)e^{j(\pi/2)} + X(2)e^{j\pi} + X(3)e^{j(3\pi/2)}\right]$$

$$= \frac{1}{4}[1 + 0 + e^{j\pi} + 0] = \frac{1}{4}[1 + 0 - 1 + 0] = 0$$

$$x(2) = \frac{1}{4} \sum_{k=0}^{3} X(k)e^{j\pi k} = \frac{1}{4}\left[X(0) + X(1)e^{j\pi} + X(2)e^{j2\pi} + X(3)e^{j3\pi}\right]$$

$$= \frac{1}{4}[1 + 0 + e^{j2\pi} + 0] = \frac{1}{4}[1 + 0 + 1 + 0] = 0.5$$

$$x(3) = \frac{1}{4} \sum_{k=0}^{3} X(k)e^{j(3\pi/2)k} = \frac{1}{4}\left[X(0) + X(1)e^{j(3\pi/2)} + X(2)e^{j3\pi} + X(3)e^{j(9\pi/2)}\right]$$

$$= \frac{1}{4}[1 + 0 + e^{j3\pi} + 0] = \frac{1}{4}[1 + 0 - 1 + 0] = 0$$

The IDFT of $X(k) = \{1, 0, 1, 0\}$ is $x(n) = \{0.5, 0, 0.5, 0\}$.

EXAMPLE 2.3 Compute the DFT of the 3-point sequence $x(n) = \{2, 1, 2\}$. Using the same sequence,compute the 6-point DFT and compare the two DFTs.

Solution: The given 3-point sequence is $x(n) = \{2, 1, 2\}$, $N = 3$.

$$\text{DFT}\, x(n) = X(k) = \sum_{n=0}^{N-1} x(n)W_N^{nk} = \sum_{n=0}^{2} x(n)e^{-j(2\pi/3)nk}, \quad k = 0, 1, 2$$

$$= x(0) + x(1)e^{-j(2\pi/3)k} + x(2)e^{-j(4\pi/3)k}$$

$$= 2 + \left(\cos\frac{2\pi}{3}k - j\sin\frac{2\pi}{3}k\right) + 2\left(\cos\frac{4\pi}{3}k - j\sin\frac{4\pi}{3}k\right)$$

When $k = 0$, $\quad X(k) = X(0) = 2 + 1 + 2 = 5$

When $k = 1$, $\quad X(k) = X(1) = 2 + \left(\cos\frac{2\pi}{3} - j\sin\frac{2\pi}{3}\right) + 2\left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}\right)$

$$= 2 + (-0.5 - j0.866) + 2(-0.5 + j0.866)$$
$$= 0.5 + j0.866$$

When $k = 2$, $\quad X(k) = X(2) = 2 + \left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}\right) + 2\left(\cos\frac{8\pi}{3} - j\sin\frac{8\pi}{3}\right)$

$$= 2 + (-0.5 + j0.866) + 2(-0.5 - j0.866)$$
$$= 0.5 - j0.866$$

$\therefore$ 3-point DFT of $x(n) = X(k) = \{5, 0.5 + j0.866, 0.5 - j0.866\}$

To compute the 6-point DFT, convert the 3-point sequence $x(n)$ into 6-point sequence bypadding with zeros.

$$x(n) = \{2, 1, 2, 0, 0, 0\}, \quad N = 6$$

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)W_N^{nk} = \sum_{n=0}^{5} x(n)e^{-j(2\pi/N)nk}, \quad k = 0, 1, 2, 3, 4, 5$$

$$= x(0) + x(1)e^{-j(2\pi/6)k} + x(2)e^{-j(4\pi/6)k} + x(3)e^{-j(6\pi/6)k} + x(4)e^{-j(8\pi/6)k}$$
$$+ x(5)e^{-j(10\pi/6)k}$$

$$= 2 + e^{-j(\pi/3)k} + 2e^{-j(2\pi/3)k}$$

When $k = 0$, $\quad X(0) = 2 + 1 + 2 = 5$

When $k = 1$, $X(1) = 2 + e^{-j(\pi/3)} + 2e^{-j(2\pi/3)}$

$$= 2 + (0.5 - j0.866) + 2(-0.5 - j0.866) = 1.5 - j2.598$$

When $k = 2$, $X(2) = 2 + e^{-j(2\pi/3)} + 2e^{-j(4\pi/3)}$

$$= 2 + (-0.5 - j0.866) + 2(-0.5 + j0.866) = 0.5 + j0.866$$

When $k = 3$, $X(3) = x(0) + x(1)e^{-j(3\pi/3)} + x(2)e^{-j(6\pi/3)}$

$$= 2 + (\cos\pi - j\sin\pi) + 2(\cos 2\pi - j\sin 2\pi)$$

$$= 2 - 1 + 2 = 3$$

When $k = 4$, $X(4) = x(0) + x(1)e^{-j(4\pi/3)} + x(2)e^{-j(8\pi/3)}$

$$= 2 + \left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}\right) + 2\left(\cos\frac{8\pi}{3} - j\sin\frac{8\pi}{3}\right)$$

$$= 2 + (-0.5 + j0.866) + 2(-0.5 - j0.866)$$

$$= 0.5 - j0.866$$

When $k = 5$, $X(5) = x(0) + x(1)e^{-j(5\pi/3)} + x(2)e^{-j(10\pi/3)}$

$$= 2 + \left(\cos\frac{5\pi}{3} - j\sin\frac{5\pi}{3}\right) + 2\left(\cos\frac{10\pi}{3} - j\sin\frac{10\pi}{3}\right)$$

$$= 2 + (0.5 - j0.866) + 2(-0.5 + j0.866) = 1.5 + j0.866$$

Tabulating the above 3-point and 6-point DFTs, we have

DFT	$X(0)$	$X(1)$	$X(2)$	$X(3)$	$X(4)$	$X(5)$
3-point	5	$0.5 + j0.866$	$0.5 - j0.866$	–	–	–
6-point	5	$1.5 - j2.598$	$0.5 + j0.866$	3	$0.5 - j0.866$	$1.5 + j0.866$

MATRIX FORMULATION OF THE DFT AND IDFT

If we let $W_N = e^{-j(2\pi/N)}$, the defining relations for the DFT and IDFT may be written as:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \qquad k = 0, 1, \ldots, N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-nk}, \qquad n = 0, 1, 2, \ldots, N-1$$

The first set of N DFT equations in N unknowns may be expressed in matrix form as:

$$\mathbf{X} = \mathbf{W_N}\mathbf{x}$$

Here $\mathbf{X}$ and $\mathbf{x}$ are $N \times 1$ matrices, and $\mathbf{W_N}$ is an $N \times N$ square matrix called the DFT matrix. The full matrix form is described by

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ W_N^0 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

THE IDFT FROM THE MATRIX FORM

The matrix **x** may be expressed in terms of the inverse of **WN** as:

$$\mathbf{x} = \mathbf{W_N^{-1}}\ \mathbf{x}$$

WN is called the IDFT matrix. We may also obtain **x** directly from the IDFT relation in matrix form, where the change of index from *n* to *k* and the change in the sign of the exponent in *e j*(2/*N*)*nk* lead tothe conjugate transpose of **WN**. We then have

$$\mathbf{x} = \frac{1}{\mathbf{N}}\left[\mathbf{W_N^*}\right]^T \mathbf{x}$$

EXAMPLE 2.4 Find the DFT of the sequence $x(n) = \{1, 2, 1, 0\}$
Solution: The DFT $X(k)$ of the given sequence $x(n) = \{1, 2, 1, 0\}$ may be obtained by solving thematrix product as follows. Here $N = 4$.

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & W_N^3 \\ W_N^0 & W_N^2 & W_N^4 & W_N^6 \\ W_N^0 & W_N^3 & W_N^6 & W_N^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -j2 \\ 0 \\ j2 \end{bmatrix}$$

The result is DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$.

EXAMPLE 2.5 Find the DFT of $x(n) = \{1, –1, 2, –2\}$.
Solution: The DFT, $X(k)$ of the given sequence $x(n) = \{1, –1, 2, –2\}$ can be determined using matrix asshown below.

$$X(k) = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1-j \\ 6 \\ -1+j \end{bmatrix}$$

$$\therefore \quad \text{DFT}\ \{x(n)\} = X(k) = \{0, -1-j, 6, -1+j\}$$

EXAMPLE 2.6. Find the 4-point DFT of $x(n) = \{1, –2, 3, 2\}$.
Solution: Given $x(n) = \{1, –2, 3, 2\}$, the 4-point DFT$\{x(n)\} = X(k)$ is determined using matrix asshown below.

$$\text{DFT}\ \{x(n)\} = X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2+j4 \\ 4 \\ -2-j4 \end{bmatrix}$$

$$\text{DFT}\ \{x(n)\} = X(k) = \{4, -2+j4, 4, -2-j4\}$$

EXAMPLE 2.6 Find the IDFT of $X(k)$={4, –j2, 0, j2} using DFT.***Solution:*** Given $X(k)$ = {4, –j2, 0, j2} _ $X^*(k)$ = {4, j2, 0, –j2} The IDFT of $X(k)$ is determined using matrix as shown below.

To find IDFT of $X(k)$ first find $X^*(k)$, then find DFT of $X^*(k)$, then take conjugate of DFT $\{X^*(k)\}$ and divide by N.

$$\text{DFT}\{X^*(k)\} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ j2 \\ 0 \\ -j2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 4 \\ 0 \end{bmatrix}$$

$$\therefore \quad \text{IDFT}[X(k)] = x(n) = \frac{1}{4}[4, 8, 4, 0]^* = \frac{1}{4}[4, 8, 4, 0] = [1, 2, 1, 0]$$

EXAMPLE 2.7 Find the IDFT of $X(k)$ = {4, 2, 0, 4} using DFT.
Solution: Given $X(k)$ = {4, 2, 0, 4}
$X^*(k)$ = {4, 2, 0, 4}
The IDFT of $X(k)$ is determined using matrix as shown below.
To find IDFT of $X(k)$, first find $X^*(k)$, then find DFT of $X^*(k)$, then take conjugate of DFT $\{X^*(k)\}$ anddivide by N

$$\text{DFT}[X^*(k)] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 4+j2 \\ -2 \\ 4-j2 \end{bmatrix}$$

$$\therefore \quad \text{IDFT}\{X(k)\} = x(n) = \frac{1}{4}[10, 4+j2, -2, 4-j2]^* = \{2.5, 1-j0.5, -0.5, 1+j0.5\}$$

EXAMPLE2.8 Find the IDFT of $X(k)$ = {1, 0, 1, 0}.
Solution: Given $X(k)$ = {1, 0, 1, 0}, the IDFT of $X(k)$, i.e. $x(n)$ is determined using matrixas shown below.

$$X^*(k) = \{1, 0, 1, 0\}^* = \{1, 0, 1, 0\}$$

$$\text{DFT}\{X^*(k)\} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\therefore \quad \text{IDFT}\{X(k)\} = x(n) = \frac{1}{4}[\text{DFT}\{X^*(k)\}]^* = \frac{1}{4}\{2, 0, 2, 0\} = \{0.5, 0, 0.5, 0\}$$

PROPERTIES OF DFT

Like the Fourier and Z-transforms, the DFT has several important properties that are used toprocess the finite duration sequences. Some of those properties are discussed as follows

Periodicity:
If a sequence $x(n)$ is periodic with periodicity of N samples, then N-point DFT of the

sequence, $X(k)$ isalso periodic with periodicity of N samples.
Hence, if $x(n)$ and $X(k)$ are an N-point DFT pair, then

$$x(n+N) = x(n) \qquad \text{for all } n$$

$$X(k+N) = X(k) \qquad \text{for all } k$$

Proof: By definition of DFT, the $(k + N)$th coefficient of $X(k)$ is given by

$$X(k+N) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi n(k+N)/N} = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} e^{-j2\pi nN/N}$$

But $e^{-j2\pi n} = 1$ for all n (Here n is an integer)

$$\therefore \qquad X(k+N) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} = X(k)$$

Linearity

If $x_1(n)$ and $x_2(n)$ are two finite duration sequences and if

$$\text{DFT } \{x_1(n)\} = X_1(k)$$

and

$$\text{DFT } \{x_2(n)\} = X_2(k)$$

Then for any real valued or complex valued constants a and b,

$$\text{DFT } \{ax_1(n) + bx_2(n)\} = aX_1(k) + bX_2(k)$$

Proof:

$$\text{DFT } \{ax_1(n) + bx_2(n)\} = \sum_{n=0}^{N-1} [ax_1(n) + bx_2(n)] e^{-j2\pi nk/N}$$

$$= a\sum_{n=0}^{N-1} x_1(n) e^{-j2\pi nk/N} + b\sum_{n=0}^{N-1} x_2(n) e^{-j2\pi nk/N}$$

$$= aX_1(k) + bX_2(k)$$

DFT of Even and Odd Sequences

The DFT of an even sequence is purely real, and the DFT of an odd sequence is purely imaginary. Therefore, DFT can be evaluated using cosine and sine

For even sequence, $X(k) = \sum_{n=0}^{N-1} x(n) \cos\left(\frac{2\pi nk}{N}\right)$

For odd sequence, $X(k) = \sum_{n=0}^{N-1} x(n) \sin\left(\frac{2\pi nk}{N}\right)$

transforms for even andodd sequences respectively.

Time Reversal of the Sequence

The time reversal of an *N*-point sequence *x*(*n*) is obtained by wrapping the

$$x[(-n), \text{ mod } N] = x(N-n), \qquad 0 \le n \le N-1$$

If DFT $\{x(n)\} = X(k)$, then

$$\text{DFT } \{x(-n), \text{mod } N\} = \text{DFT } \{x(N-n)\}$$

$$= X[(-k), \text{ mod } N] = X(N-k)$$

Proof: $\text{DFT } \{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n) e^{-j2\pi nk/N}$

sequence $x(n)$ around the circle in the clockwise direction. It is denoted as $x[(-n)$, mod $N]$ and

Changing index from n to m, where $m = N - n$, we have $n = N - m$.

Now,

$$\text{DFT}\left\{x(N-n)\right\} = \sum_{m=0}^{N-1} x(m)e^{-j2\pi k(N-m)/N}$$

$$= \sum_{m=0}^{N-1} x(m)e^{-j(2\pi/N)kN}e^{j(2\pi/N)km}$$

$$= \sum_{m=0}^{N-1} x(m)e^{j(2\pi/N)km}$$

$$= \sum_{m=0}^{N-1} x(m)e^{j(2\pi/N)km}e^{-j2\pi m}$$

$$= \sum_{m=0}^{N-1} x(m)e^{-j2\pi m[(N-k)/N]} = X(N-k)$$

Circular Frequency Shift

If $\quad \text{DFT}\ \{x(n)\} = X(k)$

Then, $\quad \text{DFT}\ \{x(n)\ e^{j2\pi ln/N}\} = X\{(k-l), (\text{mod } N)]$

Proof:

$$\text{DFT}\ \{x(n)e^{j2\pi l n/N}\} = \sum_{n=0}^{N-1} x(n)e^{j2\pi l n/N}e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} x(n)e^{-j2\pi n(k-l)/N}$$

$$= \sum_{n=0}^{N-1} x(n)e^{-j2\pi n(N+k-l)/N}$$

$$= X(N+k-l) = X[(k-l), (\text{mod } N)]$$

Complex Conjugate Property

If $\quad \text{DFT}\{x(n)\} = X(k)$

Then $\quad \text{DFT}\{x^*(n)\} = X^*(N-k) = X^*[(-k), \text{mod } N]$

Proof:
$$\text{DFT}\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n)\, e^{-j2\pi kn/N}$$
$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{j2\pi nk/N}\right]^* = \left[\sum_{n=0}^{N-1} x(n) e^{-j2\pi n(N-k)/N}\right]^* = X^*(N-k)$$

$$\text{DFT}\{x^*(N-n)\} = X^*(k)$$

Proof:
$$\text{IDFT}\{X^*(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k) e^{j2\pi kn/N}$$
$$= \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k) e^{-j2\pi kn/N}\right]^* = \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k) e^{j2\pi k(N-n)/N}\right]^* = x^*(N-n)$$

DFT of Delayed Sequence (Circular time shift of a sequence)

Let $x(n)$ be a discrete sequence, and $x'(n)$ be a delayed or shifted sequence of $x(n)$ by n_0 units of time.

If $\quad \text{DFT}\{x(n)\} = X(k)$

Then, $\quad \text{DFT}\{x'(n)\} = \text{DFT}\{x[(n-n_0), \text{mod } N]\} = X(k)\, e^{-j2\pi n_0 k/N}$

Proof: By the definition of IDFT,

$$\text{IDFT}\{X(k)\} = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}nk}$$

Replacing n by $n - n_0$, we have

$$x(n-n_0) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}(n-n_0)k}$$
$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[X(k)\, e^{-j\frac{2\pi}{N}n_0 k}\right] e^{j\frac{2\pi}{N}nk}$$
$$= \text{IDFT}\left[X(k)\, e^{-j\frac{2\pi}{N}n_0 k}\right]$$

On taking DFT on both sides, we get

$$\text{DFT}\,[x(n-n_0)] = X(k)\, e^{-j\frac{2\pi}{N}kn_0}$$

DFT of Real Valued Sequences

Let $x(n)$ be a real sequence. By definition of DFT,

$$\text{DFT }\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} x(n)\left(\cos\frac{2\pi}{N}nk - j\sin\frac{2\pi}{N}nk \right)$$

$$= \sum_{n=0}^{N-1} x(n)\cos\frac{2\pi}{N}nk - j\sum_{n=0}^{N-1} x(n)\sin\frac{2\pi}{N}nk$$

Also $X(k) = X_R(k) + jX_I(k)$

Therefore, we can say

Real part $\quad X_R(k) = \sum_{n=0}^{N-1} x(n)\cos\left(\frac{2\pi}{N}nk\right), \quad \text{for } 0 \le k \le N-1$

Imaginary part $\quad X_I(k) = -\sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi}{N}nk\right), \quad \text{for } 0 \le k \le N-1$

When $x(n)$ is real, then $X(k)$ will have the following features:

(a) $X(k)$ has complex conjugate symmetry, i.e. $X(k) = X^*(N-k)$
(b) Real component is even function, i.e. $X_R(k) = X_R(N-k)$
(c) Imaginary component is odd function, i.e. $X_I(k) = -X_I(N-k)$
(d) Magnitude function is even function, i.e. $|X(k)| = |X(N-k)|$
(e) Phase function is odd function, i.e. $\angle X(k) = -\angle X(N-k)$
(f) If $x(n) = x(-n)$ (even sequence), then $X(k)$ is purely real.
(g) If $x(n) = -x(-n)$ (odd sequence), then $X(k)$ is purely imaginary.

Multiplication of Two Sequences

If $\quad \text{DFT }[x_1(n)] = X_1(k)$

and $\quad \text{DFT }[x_2(n)] = X_2(k)$

Then $\quad \text{DFT}[x_1(n)x_2(n)] = \frac{1}{N}[X_1(k) \oplus X_2(k)]$

Circular Convolution of Two Sequences

The convolution property of DFT says that, the multiplication of DFTs of two sequences is equivalent to the DFT of the circular convolution of the two sequences.

Let DFT $[x_1(n)] = X_1(k)$ and DFT $[x_2(n)] = X_2(k)$, then by the convolution property $X_1(k)X_2(k) = \text{DFT}\{x_1(n) \circledast x_2(n)\}$.

Proof: Let $x_1(n)$ and $x_2(n)$ be two finite duration sequences of length N. The N-point DFTs of the two sequences are:

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi}{N}nk}, \quad k = 0, 1, \ldots, N-1$$

$$X_2(k) = \sum_{l=0}^{N-1} x_2(l) e^{-j\frac{2\pi}{N}lk}, \quad k = 0, 1, \ldots, N-1$$

On multiplying the above two DFTs, we obtain the result as another DFT, say, $X_3(k)$. Now, $X_3(k)$ will be N-point DFT of a sequence $x_3(m)$.

$$\therefore \quad X_3(k) = X_1(k)X_2(k) \quad \text{and} \quad \text{IDFT}\{X_3(k)\} = x_3(m)$$

By the definition of IDFT,

$$x_3(m) = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)\, e^{j\frac{2\pi}{N}mk}, \quad m = 0, 1, 2, \ldots, N-1$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)X_2(k)\, e^{j\frac{2\pi}{N}mk}$$

Using the above equations for $X_1(k)$ and $X_2(k)$, the equation for $x_3(m)$ is:

$$x_3(m) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi}{N}nk} \sum_{l=0}^{N-1} x_2(l) e^{-j\frac{2\pi}{N}lk} \, e^{j\frac{2\pi}{N}mk}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(m-n-l)}$$

Let $m - n - l = PN$ where P is an integer.

$$\therefore \quad e^{j\frac{2\pi}{N}k(m-n-l)} = e^{j\frac{2\pi}{N}kPN} = e^{j2\pi kP} = (e^{j2\pi P})^k$$

We know that

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi k}{N}(m-n-l)} = \sum_{k=0}^{N-1} (e^{j2\pi P})^k = \sum_{k=0}^{N-1} 1^k = \sum_{k=0}^{N-1} 1 = N$$

Therefore, the above equation for $x_3(m)$ can be written as:

$$x_3(m) = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l)\, N = \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l)$$

If $x_2(l)$ is a periodic sequence with periodicity of N samples, then $x_2(l \pm PN) = x_2(l)$

Here $\quad m - n - l = PN$

$\therefore \quad l = m - n - PN$

$\therefore \quad x_2(l) = x_2(m - n - PN) = x_2(m - n) = x_2[(m - n), \text{mod } N]$

Therefore, $x_3(m)$ can be

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n) \sum_{n=0}^{N-1} x_2(m-n) = \sum_{n=0}^{N-1} x_1(n)\, x_2(m-n)$$

Replacing m by n and n by k, we have $x_3(n) = \sum_{k=0}^{N-1} x_1(k)\, x_2(n-k)$

Note: For simplicity, $x_2[(m - n), \text{mod } N]$ is represented as $x_2(m - n)$.

The equation for $x_2(l)$ is in the form of convolution sum. Since the equation for $x_2(l)$ involves the index $[(m - n), \text{mod } N]$, it is called circular convolution.

Hence, we conclude that multiplication of the DFTs of two sequences is equivalent to the DFT of the circular convolution of the two sequences.

$$X_1(k)X_2(k) = \text{DFT}\{x_1(n) \circledast x_2(n)\}$$

Parseval's Theorem

Parseval's theorem says that the DFT is an energy-conserving transformation and allows usto find the signal energy either from the signal or its spectrum. This implies that the sum ofsquares of the signal samples is related to the sum of squares

If $\quad \text{DFT}\{x_1(n)\} = X_1(k)$

and $\quad \text{DFT}\{x_2(n)\} = X_2(k)$

Then $\quad \sum_{n=0}^{N-1} x_1(n)\, x_2^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\, X_2^*(k)$

of the magnitude of the DFT samples.

Circular Correlation

For complex valued sequences $x(n)$ and $y(n)$,

If $\quad \text{DFT}\{x(n)\} = X(k)$

and $\quad \text{DFT}\{y(n)\} = Y(k)$

Then $\quad \text{DFT}\{r_{xy}(l)\} = \text{DFT}\left[\sum_{n=0}^{N-1} x(n)\, y^*((n-l), \text{mod } N)\right] = X(k)Y^*(k)$

where, $r_{xy}(l)$ is the circular cross correlation sequence. The properties of DFT are summarized in Table 6.3.

Linear Convolution using DFT

The DFT supports only circular convolution. When two numbers of N-point sequence are circularly convolved, it produces another N-point sequence. For circular convolution, one of the sequence should be periodically extended. Also the resultant sequence is periodic with period N. The linear convolution of two sequences of length $N1$ and $N2$ produces an output sequence of length $N1 + N2 – 1$. To perform linear convolution using DFT, both the sequences should be converted to $N1 + N2 – 1$ sequences by padding with zeros. Then take $N1 + N2 – 1$-point DFT of both the sequences and determine the product of their DFTs. The resultant sequence is given by the IDFT of the product of DFTs. [Actually the response is given by the circular convolution of the $N1 + N2 – 1$ sequences]. Let $x(n)$ be an $N1$-point sequence and $h(n)$ be an $N2$-point sequence. The linear convolution of $x(n)$ and $h(n)$ produces a sequence $y(n)$ of length $N1 + N2 – 1$. So pad $x(n)$ with $N2 – 1$ zeros and $h(n)$ with $N1 – 1$ zeros and make both of them of length $N1 + N2 – 1$. Let $X(k)$ be an $N1 + N2 – 1$-point DFT of $x(n)$, and $H(k)$ be an $N1 + N2 – 1$-point DFT of $h(n)$. Now, the sequence $y(n)$ is given by the inverse DFT of the product $X(k)$ $H(k)$.

$$y(n) = \text{IDFT}\ \{X(k)H(k)\}$$

This technique of convolving two finite duration sequences using DFT techniques is called fast convolution. The convolution of two sequences by convolution sum formula. This technique of convolving two finite duration sequences using DFT techniques is called fast convolution. The convolution of two sequences by convolution sum formula.

$$\text{Y(n)}=\sum_{-\infty}^{\infty} x(k)h(n-k)$$

is called direct convolution or slow convolution. The term fast is used because the DFT can be evaluated rapidly and efficiently using any of a large class of algorithms called Fast Fourier Transform (FFT). In a practical sense, the size of DFTs need not be restricted to $N1 + N2 – 1$-point transforms.

Any number L can be used for the transform size subject to the restriction L _ $(N1 + N2 – 1)$. If $L > (N1 + N2 – 1)$, then $y(n)$ will have zero valued samples at the end of the period.

EXAMPLE 2.1 Find the linear convolution of the sequences $x(n)$ and $h(n)$ using DFT.

$x(n) = \{1, 2\}$, $h(n) = \{2, 1\}$

Solution: Let $y(n)$ be the linear convolution of $x(n)$ and $h(n)$. $x(n)$ and $h(n)$ are of length 2 each. So the linear convolution of $x(n)$ and $h(n)$ will produce a 3 sample sequence $(2 + 2 – 1 = 3)$. To avoid time aliasing, we convert the 2 sample input sequences into 3 sample sequences by padding with zeros.

$x(n) = \{1, 2, 0\}$ and $h(n) = \{2, 1, 0\}$

By the definition of N-point DFT, the 3-point DFT of $x(n)$ is:

$$X(k) = \sum_{n=0}^{2} x(n)\, e^{-j\frac{2\pi}{3}kn} = x(0)e^{0} + x(1)e^{-j\frac{2\pi}{3}k} + x(2)e^{-j\frac{4\pi}{3}k} = 1 + 2\, e^{-j\frac{2\pi}{3}k}$$

When $k = 0$, $X(0) = 1 + 2e^{0} = 3$

When $k = 1$, $X(1) = 1 + 2e^{-j\frac{2\pi}{3}} = 1 + 2(-0.5 - j0.866) = -j1.732$

When $k = 2$, $X(2) = 1 + 2e^{-j\frac{4\pi}{3}} = 1 + 2(-0.5 + j0.866) = j1.732$

By the definition of N-point DFT, the 3-point DFT of $h(n)$ is:

$$H(k) = \sum_{n=0}^{2} h(n)\, e^{-j\frac{2\pi}{3}nk} = h(0)e^{0} + h(1)e^{-j\frac{2\pi}{3}k} + h(2)e^{-j\frac{4\pi}{3}k} = 2 + e^{-j\frac{2\pi}{3}k}$$

When $k = 0$, $H(0) = 2 + 1 = 3$

When $k = 1$, $H(1) = 2 + e^{-j\frac{2\pi}{3}} = 2 + (-0.5 - j0.866) = 1.5 - j0.866$

When $k = 2$, $H(2) = 2 + e^{-j\frac{4\pi}{3}} = 2 + (-0.5 + j0.866) = 1.5 + j0.866$

Let $Y(k) = X(k)\,H(k)$ for $k = 0, 1, 2$

When $k = 0$, $Y(0) = X(0)H(0) = (3)(3) = 9$

When $k = 1$, $Y(1) = X(1)H(1) = (-j1.732)(1.5 - j0.866) = -1.5 - j2.598$

When $k = 2$, $Y(2) = X(2)H(2) = (j1.732)(1.5 + j0.866) = -1.5 + j2.598$

$\therefore$ $Y(k) = \{9,\ -1.5 - j2.598, -1.5 + j2.598\}$

The sequence $y(n)$ is obtained from IDFT of $Y(k)$. By definition of IDFT,

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\, e^{j\frac{2\pi}{N}nk}; \quad \text{for} \quad n = 0, 1, 2, \dots, N-1$$

$$y(n) = \frac{1}{3}\sum_{k=0}^{2} Y(k) e^{j\frac{2\pi}{3}nk} = \frac{1}{3}\left[Y(0)e^{0} + Y(1)e^{j\frac{2\pi}{3}n} + Y(2)e^{j\frac{4\pi}{3}n}\right] \quad \text{for} \quad n = 0, 1, 2$$

$$\text{When } n = 0,\ y(0) = \frac{1}{3}[Y(0) + Y(1) + Y(2)]$$

$$= \frac{1}{3}[9 + (-1.5 - j2.598) + (-1.5 + j2.598)]$$

$$= \frac{1}{3}[6] = 2$$

$$\text{When } n = 1,\ y(1) = \frac{1}{3}\left[Y(0) + Y(1)\,e^{j\frac{2\pi}{3}} + Y(2)\,e^{j\frac{4\pi}{3}}\right]$$

$$= \frac{1}{3}[9 + (-1.5 - j2.598)(-0.5 + j0.866) + (-1.5 + j2.598)(-0.5 - j0.866)]$$

$$= \frac{1}{3}[9 + 0.75 + 2.25 + 0.75 + 2.25] = 5$$

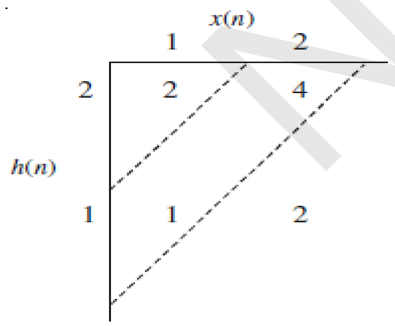
$$\text{When } n = 2,\ y(2) = \frac{1}{3}\left[Y(0) + Y(1)\,e^{j\frac{4\pi}{3}} + Y(2)\,e^{j\frac{8\pi}{3}}\right]$$

$$= \frac{1}{3}[9 + (-1.5 - j2.598)(-0.5 + j0.866) + (-1.5 + j2.598)(-0.5 - j0.866)]$$

$$= \frac{1}{3}[9 + 0.75 - 2.25 + 0.75 - 2.25] = 2$$

$$\therefore \qquad y(n) = \{2, 5, 2\}$$

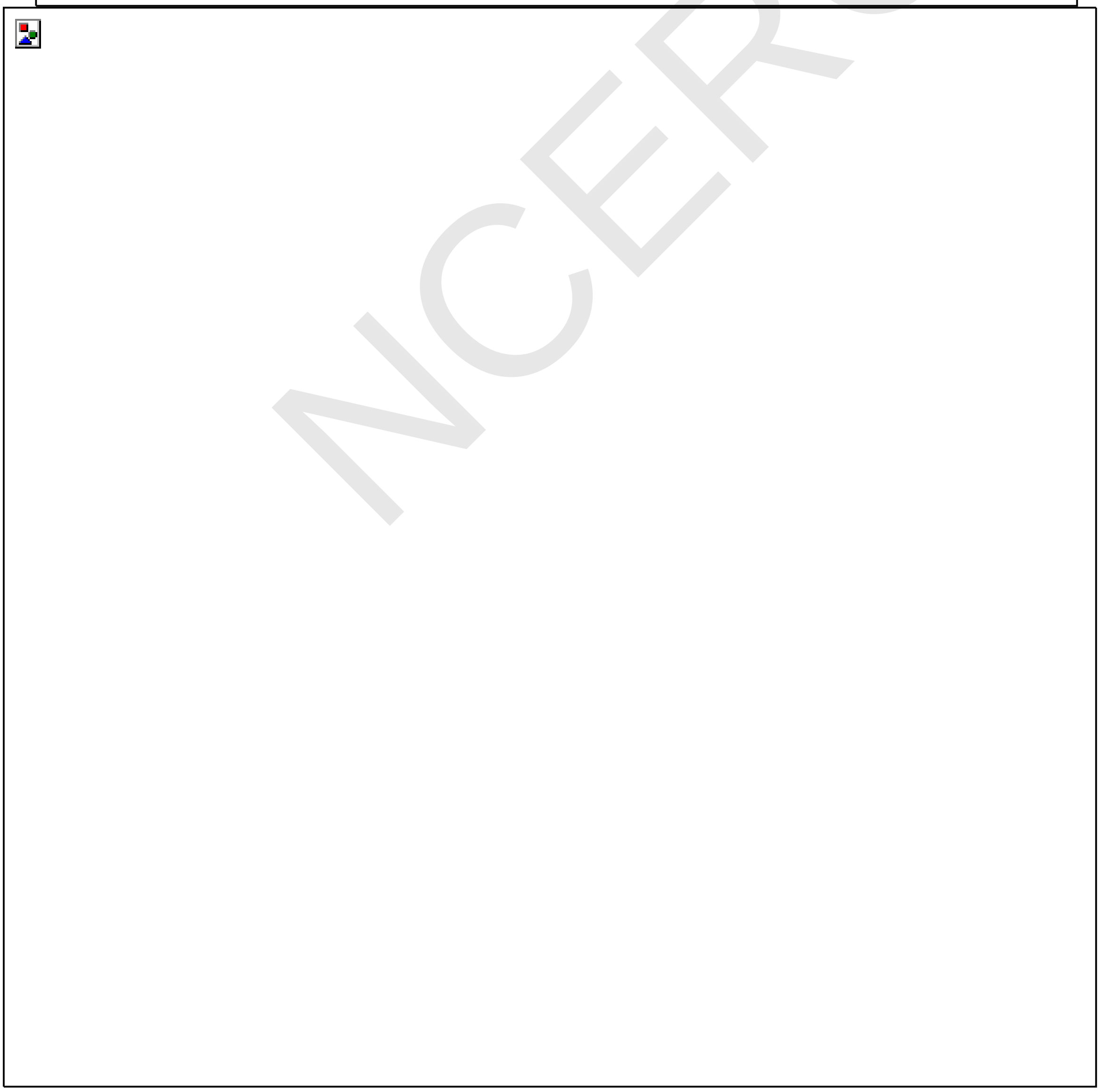
The linear convolution of $x(n) = \{1, 2\}$ and $h(n) = \{2, 1\}$ is obtained using the tabularmethod as shown below.



From the above table, $y(n) = \{2, 1 + 4, 2\} = \{2, 5, 2\}$.

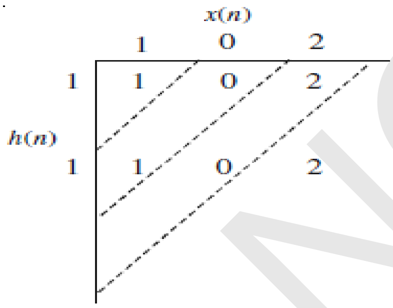
EXAMPLE 2.2 Find the linear convolution of the sequences $x(n)$ and $h(n)$ using DFT. $x(n) = \{1, 0, 2\}$, $h(n) = \{1, 1\}$

Solution: Let $y(n)$ be the linear convolution of $x(n)$ and $h(n)$. $x(n)$ is of length 3 and $h(n)$ isof length 2. So the linear convolution of $x(n)$ and $h(n)$ will produce a 4-sample sequence
$(3 + 2 – 1 = 4)$. To avoid time aliasing, we convert the 2-sample and 3-sample sequencesinto 4-sample sequences by padding with zeros.
$x(n) = \{1, 0, 2, 0\}$ and $h(n) = \{1, 1, 0, 0\}$
By the definition of N-point DFT, the 4-point DFT of $x(n)$ is:



Therefore, the linear convolution of $x(n)$ and $h(n)$ is:
$y(n) = x(n) * h(n) = \{1, 1, 2, 2\}$

The linear convolution of $x(n) = \{1, 0, 2\}$ and $h(n) = \{1, 1\}$ is obtained using the tabularmethod as shown below.



From the above table, $y(n) = \{1, 1, 2, 2\}$.

OVERLAP-ADD METHOD :

In overlap-add method, the longer sequence $x(n)$ of length L is split into m number of smaller sequences of length N equal to the size of the smaller sequence $h(n)$. (If required zero padding may be done to L so that $L = mN$). The linear convolution of each section (of length N) of longer sequence with the smaller sequence of length N is performed. This gives an output sequence of length $2N – 1$.
In t his method, the last $N – 1$ samples of each output sequence overlaps with the first $N – 1$ samples of next section. While combining the output sequences of the various sectioned convolutions, the corresponding samples of overlapped regions are added and the samples of non-overlapped regions are retained as such. If the linear convolution is to be performed by DFT (or FFT), since DFT supports only circular convolution and not linear convolution directly, we have to pad each section of the longersequence (of length N) and also the smaller sequence (of length N) with $N – 1$ zeros before computing the circular convolution of each section with the smaller sequence. The steps for this fast convolution by overlap-add method are as follows:

***Step* 1**: $N - 1$ zeros are padded at the end of the impulse response sequence $h(n)$ which isof length N and a sequence of length $2N - 1$ is obtained. Then the $2N - 1$ point FFT is performed and the output values are stored.

***Step* 2**: Split the data, i.e. $x(n)$ into m blocks each of length N and pad $N - 1$ zeros to each

***Step* 3**: The stored frequency response of the filter, i.e. the FFT output sequence obtainedin Step 1 is multiplied by the FFT output sequence of each of the selected block in Step 2.
***Step* 4**: A $2N – 1$ point inverse FFT is performed on each product sequence obtained in Step 3.
***Step* 5**: The first $(N – 1)$ IFFT values obtained in Step 4 for each block, overlapped with the last $N – 1$ values of the previous block. Therefore, add the overlapping values and keep the non-overlapping values as they are. The result is the linear convolution of $x(n)$ and $h(n)$.

OVERLAP-SAVE METHOD

In overlap-save method, the results of linear convolution of the various sections are obtainedusing circular convolution. Let $x(n)$ be a longer sequence of length L and $h(n)$ be a smaller sequence of length N. The regular convolution of sequences of length L and N has $L + N – 1$samples. If $L > N$, we have to zero pad the second sequence $h(n)$ to length L. So their linear convolution will have $2L – 1$ samples. Its first $N – 1$ samples are contaminated by

wraparound and the rest corresponds to the regular convolution. To understand this let L =12 and $N = 5$. If we pad N by 7 zeros, their regular convolution has 23 (or $2L – 1$) samples with 7 trailing zeros ($L – N = 7$). For periodic convolution, 11 samples ($L – 1 = 11$) are wrapped around. Since the last 7 (or $L – N$) are zeros only, first four samples $(2L – 1) – (L)$
$– (L – N) = N – 1 = 5 – 1 = 4$ of the periodic convolution are contaminated by wraparound. This idea is the basis of overlap-save method. First, we add $N – 1$ leading zeros to the longersequence $x(n)$ and section it into k overlapping (by $N – 1$) segments of length M. Typically
we choose $M = 2N$. Next, we zero pad $h(n)$ (with trailing zeros) to length M, and find the periodic convolution of $h(n)$ with each section of $x(n)$. Finally, we discard the first $N – 1$ (contaminated) samples from each convolution and glue (concatenate) the results to give the required convolution.
***Step* 1**: N zeros are padded at the end of the impulse response $h(n)$ which is of length N and a sequence of length $M = 2N$ is obtained. Then the $2N$ point FFT is performed and the output values are stored.
***Step* 2**: A $2N$ point FFT on each selected data block is performed. Here each data block begins with thelast $N – 1$ values in the previous data block, except the first data block which begins with $N – 1$ zeros.
***Step* 3**: The stored frequency response of the filter, i.e. the FFT output sequence obtained in Step 1 is multiplied by the FFT output sequence of each of the selected blocks obtained in Step 2.
***Step* 4**: A $2N$ point inverse FFT is performed on each of the product sequences obtained inStep 3.
***Step* 5**: The first $N – 1$ values from the output of each block are discarded and the remaining values arestored. That gives the response $y(n)$.
In either of the above two methods, the FFT of the shorter sequence need be found only once, stored, and reused for all subsequent partial convolutions. Both methods allow online implementation if we can tolerate a small processing delay that equals the time required for each section of the long sequenceto arrive at the processor

Fast Fourier Transform

2.2 INTRODUCTION

The N-point DFT of a sequence $x(n)$ converts the time domain N-point sequence $x(n)$ to a frequency domain N-point sequence $X(k)$. The direct computation of an N-point DFT requires N x N complex multiplications and $N(N - 1)$ complex additions. Many methods were developed for reducing the number of calculations involved. The most popular of these is the Fast Fourier Transform (FFT), a method developed by Cooley and Turkey. The FFT may be defined as an algorithm (or a method) for computing the DFT efficiently (with reduced number of calculations). The computational efficiency is achieved by adopting a divide and conquer approach. This approach is based on the decomposition of an N-point DFT into successively smaller DFTs and then combining them to give the total transform. Based onthis basic approach, a family of computational algorithms were developed and they are collectively known as FFT algorithms. Basically there are two FFT algorithms; Decimation- in- time (DIT) FFT algorithm and Decimation-in-frequency (DIF) FFT algorithm. In this chapter, we discuss DIT FFT and DIF FFT algorithms and the computation of DFT by these methods.

FAST FOURIER TRANSFORM

The DFT of a sequence $x(n)$ of length N is expressed by a complex-valued sequence $X(k)$ as

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}, \quad K = 0,1,2,....N-1 \text{ where}$$

Let W_N be the complex valued phase factor, which is an N^{th} root of unity given by

$$W_N = e^{-j2\pi nk/N}$$

Thus,

$X(k)$ becomes,

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \quad K = 0,1,2, \ldots N-1$$

Similarly, IDFT is written as

$$x(n) = \sum_{n=0}^{N-1} X(K) W_N^{-nk}, \quad n = 0,1,2,...N-1$$

From the above equations for $X(k)$ and $x(n)$, it is clear that for each value of k, the direct computation of $X(k)$ involves N complex multiplications ($4N$ real multiplications) and $N - 1$ complex additions ($4N - 2$ real additions). Therefore, to compute all N values of DFT, N^2 complex multiplications and $N(N - 1)$ complex additions are required. In fact the DFT and IDFT involve the same type of computations.

If $x(n)$ is a complex-valued sequence, then the N-point DFT given in equation for $X(k)$ can be expressed as

$$X(k) = X_R(k) + jX_I(k)$$

The direct computation of the DFT needs $2N^2$ evaluations of trigonometric functions, $4N^2$ real multiplications and $4N(N - 1)$ real additions. Also this is primarily inefficient as it cannot exploit the symmetry and periodicity properties of the phase factor W_N, which are

Symmetry property $\quad W_N^{k+N/2} = -W_N^K$

Periodicity property $\quad W_N^{k+N} = W_N^K$

FFT algorithm exploits the two symmetry properties and so is an efficient algorithm for DFT computation.

By adopting a divide and conquer approach, a computationally efficient algorithm can be developed. This approach depends on the decomposition of an N-point DFT into successively smaller size DFTs. An N-point sequence, if N can be expressed as $N = r_1 r_2 r_3, ..., r_m$. where $r_1 = r_2 = r_3 = ... = r_m$, then $N = r^m$, can be decimated into r-point sequences. For each r- point sequence, r-point DFT can be computed. Hence the DFT is of size r. The number r is called the radix of the FFT algorithm and the number m indicates the number of stages in computation. From the results of r-point DFT, the r^2-point DFTs are computed. From the results of r^2-point DFTs, the r^3-point DFTs are computed and so on, until we get r^m-point DFT. If $r = 2$, it is called radix-2 FFT.

DECIMATION IN TIME (DIT) RADIX-2 FFT

In Decimation in time (DIT) algorithm, the time domain sequence $x(n)$ is decimated and smaller point DFTs are computed and they are combined to get the result of N-point DFT.

In general, we can say that, in DIT algorithm the N-point DFT can be realized from two numbers of $N/2$-point DFTs, the $N/2$-point DFT can be realized from two numbers of $N/4$- point DFTs, and so on.

In DIT radix-2 FFT, the N-point time domain sequence is decimated into 2-point sequences and the 2-point DFT for each decimated sequence is computed. From the results of 2-point DFTs, the 4-point DFTs, from the results of 4-point DFTs, the 8-point DFTs andso on are computed until we get N-point DFT.

For performing radix-2 FFT, the value of r should be such that, $N = 2^m$. Here, the decimation can be performed m times, where $m = \log_2 N$. In direct computation of N- point DFT, the total number of complex additions are $N(N - 1)$ and the total number of complex multiplications are N^2. In radix-2 FFT, the total number of complex additions are reduced to $N \log_2 N$ and the total number of complex multiplications are reduced to $(N/2) \log_2 N$.

Let $x(n)$ be an N-sample sequence, where N is a power of 2. Decimate or break this sequence into two sequences $f_1(n)$ and $f_2(n)$ of length $N/2$, one composed of the even indexed values of $x(n)$ and the other of odd indexed values of $x(n)$.

Given sequence $\quad x(n): x(0), x(1), x(2), \ldots\ldots x(\frac{N}{2} - 1) \; x(N - 1)$

Even indexed sequence

Odd indexed sequence

$$f_1(n) = x(2n) : x(0), x(2), x(4), \ldots x(N-2)$$

$$f_2(n) = x(2n+1) : x(1), x(3), x(5), \ldots x(N-1)$$

We know that the transform X(k) of the N-point sequence x(n) is given by

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \quad K = 0,1,2,...N-1.$$

$$X(K) = \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + \sum_{n=N/2}^{N-1} x(n) W_N^{nk}$$

$$X(K) = \sum_{n=even}^{N-1} x(n) W_N^{nk} + \sum_{n=odd}^{N-1} x(n) W_N^{nk}$$

When n is replaced by 2n, the even numbered samples are selected and when n is replaced by2n + 1, the odd numbered samples are selected. Hence,

$$X(K) = \sum_{n=0}^{N/2-1} x(2n) W_N^{2nk} + \sum_{n=0}^{N/2-1} x(2n+1) W_N^{(2n+1)k}$$

Rearranging each part of X(k) into (N/2)-point transforms using

$$W_N^{2nk} = (W_N^2)^{nk} = \left[e^{-j\frac{2\pi}{N}} \right]^{2nk} = W_{N/2}^{nk} \quad \text{and} \quad W_N^{(2n+1)k} = (W_N^k) W_{N/2}^{nk}$$

We can write

$$X(K) = \sum_{n=0}^{N/2-1} f_1(n) W_{N/2}^{nk} + W_N^k \sum_{n=0}^{N/2-1} f_2(n) W_{N/2}^{nk}$$

By definition of DFT, the N/2-point DFT of f1(n) and f2(n) is given by

$$F_1(K) = \sum_{n=0}^{N/2-1} f_1(n) W_{N/2}^{nk} \quad \& \quad F_2(K) = \sum_{n=0}^{N/2-1} f_2(n) W_{N/2}^{nk}$$

$$X(k) = F_1(K) + W_N^k F_2(K), ...k = 0,1,2,3, \; N-1$$

The implementation of this equation for X(k) is shown in the following Figure . This first step in the decomposition breaks the N-point transform into two (N/2)-point transforms and the k WN provides the N-point combining algebra. The DFT of a sequence is periodic with period given bythe number of points of DFT. Hence, F1(k) and F2(k) will be periodic with period N/2.

$$F_1(k + N/2) = F_1(K), \& F_2(k + N/2) = F_2(K)$$

$$F_1(k + N/2) = F_1(K), \& F_2(k + N/2) = F_2(K)$$

In addition, the phase factor $W_N^{(k + N/2)} = -(W_N^k)$

Therefore, for k ≥ N/2, X(k) is given by

$$X(K) = F_1(k - N/2) - W_N^k F_2(K - N/2)$$

The implementation using the periodicity property is also shown in following Figure

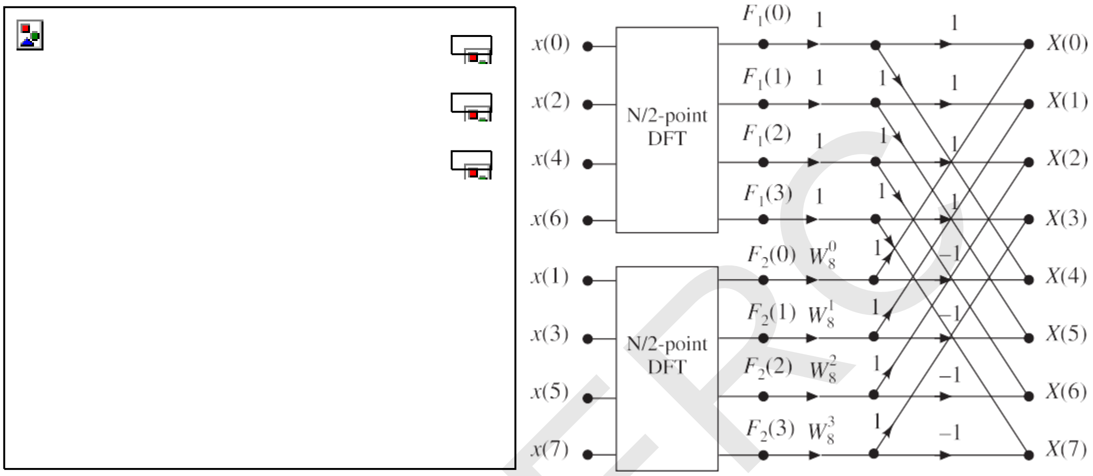


Figure 2.1 Illustration of flow graph of the first stage DIT FFT algorithm for $N = 8$.

Having performed the decimation in time once, we can repeat the process for each of the sequences $f1(n)$ and $f2(n)$. Thus $f1(n)$ would result in two $(N/4)$-point sequences and $f2(n)$ would result in another two $(N/4)$-point sequences.

THE 8-POINT DFT USIKG RADIX-2 DIT FFT

The computation of 8-point DFT using radix-2 FFT involves three stages of computation. Here $N = 8 = 2^3$, therefore, $r = 2$ and $m = 3$. The given 8-point sequence is decimated into four 2-point sequences. For each 2-point sequence, the two point DFT is computed. From the results of four 2-point DFTs, two 4-point DFTs are obtained and from the results of two 4- point DFTs, the 8-point DFT is obtained.

Let the given 8-sample sequence $x(n)$ be $\{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$. The 8-samples should be decimated into sequences of two samples. Before decimation they are arranged in bit reversed order as shown in Table 2.1.

Figure 2.4 Illustration of complete flow graph obtained by combining all the three stages for $N = 8$.

TABLE 2.1 Normal and bit reversed order for $N = 8$.

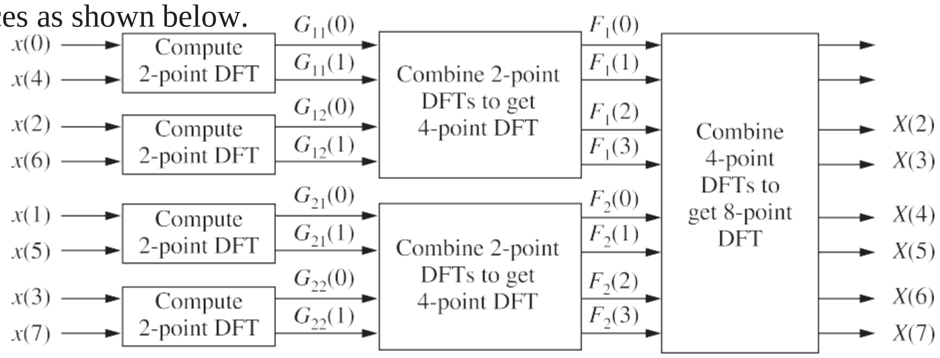
Normal order		*Bit reversed order*	
$x(0)$	$x(000)$	$x(0)$	$x(000)$
$x(1)$	$x(001)$	$x(4)$	$x(100)$
$x(2)$	$x(010)$	$x(2)$	$x(010)$
$x(3)$	$x(011)$	$x(6)$	$x(110)$
$x(4)$	$x(100)$	$x(1)$	$x(001)$
$x(5)$	$x(101)$	$x(5)$	$x(101)$
$x(6)$	$x(110)$	$x(3)$	$x(011)$
$x(7)$	$x(111)$	$x(7)$	$x(111)$

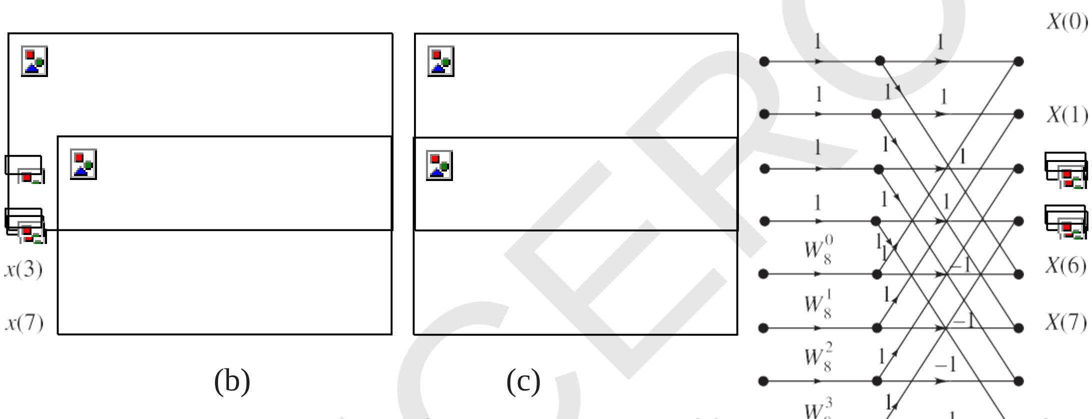
The $x(n)$ in bit reversed order is decimated into 4 numbers of 2-point sequences asshown below.

(i) $x(0)$ and $x(4)$
(ii) $x(2)$ and $x(6)$
(iii) $x(1)$ and $x(5)$
(iv) $x(3)$ and $x(7)$

Using the decimated seuences as input, the 8-point DFT is computed. Figure 7.5 shows thethree stages of computation of an 8-point DFT.

The computation of 8-point DFT of an 8-point sequence in detail is given below. The 8-point sequence is decimated into 4-point sequences and 2-point sequences as shown below.





(a) (b) (c)

Figure 7.6 (a)–(c) Flow graphs for implementation of first, 2nd and 3rd stages of computation.

Butterfly Diagram

Observing the basic computations performed at each stage, we can arrive at the followingconclusions:

(i) In each computation, two complex numbers *a* and *b* are considered.
(ii) The complex number *b* is multiplied by a phase factor W^k_N.
(iii) The product bW^k_N is added to the complex number *a* to form a new complex number *A*.
(i) The product bW^k_N is subtracted from complex number *a* to form new complex number *B*.

The above basic computation can be expressed by a signal flow graph shown in Figure 7.7.The signal flow graph is also called butterfly diagram since it resembles a butterfly.

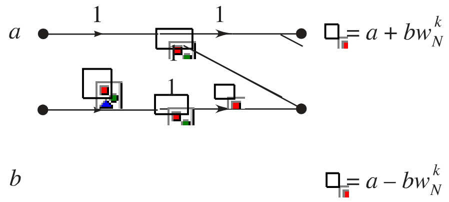


Figure 7.7 Basic butterfFy diagram or fFow graph of radix–2 DI† FF†.

The complete flow graph for 8-point DIT FFT considering periodicity drawn

2 4 8

in a way to remember easily is shown in Figure 7.8. In radix-2 FFT, $N/2$ butterflies per stage are required to represent the computational process. In the butterfly diagram for 8-point DFT shown in Figure 7.8, for symmetry, W_2^0, W_4^0 and W_8^0 are shown on the graph eventhough they are unity. The subscript 2 indicates that it is the first stage of computation. Similarly, subscripts 4 and 8 indicate the second and third stages of computation.

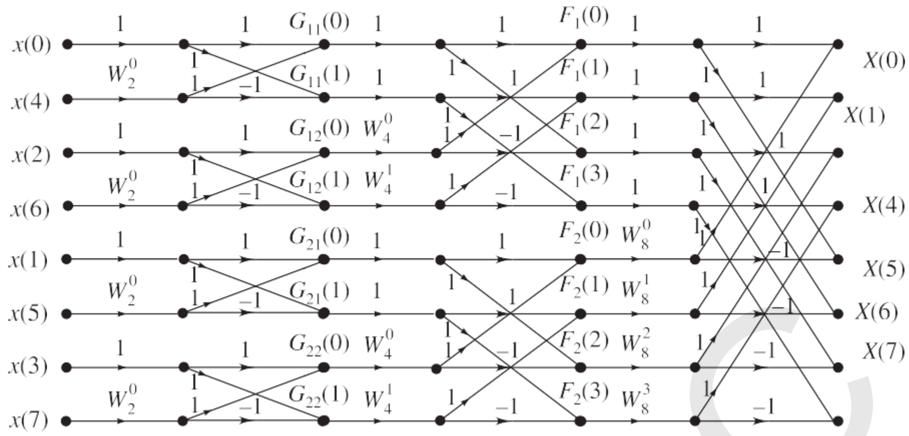


Figure 7.8 The signal flow graph or butterfly diagram for 8–point radix–2 DIT FFT.

$$X(K) = \sum_{n=0}^{N-1} x(n)W_N^{nk} = \sum_{n=0}^{N/2-1} x(n)W_N^{nk} + \sum_{n=N/2}^{N-1} x(n)W_N^{nk}$$

$$= \sum_{n=0}^{N/2-1} x(n)W_N^{nk} + \sum_{n=0}^{N/2-1} x(n+N/2)W_N^{(n+N/2)k}$$

It is important to observe that while the above equation for $X(k)$ contains two summations over $N/2$-points, each of these summations is not an $N/2$-point DFT, since W_N^{nk} rather than $W_{N/2}^{nk}$

$$X(K) = \sum_{n=0}^{N/2-1} x(n)W_N^{nk} + W_N^{(N/2)k} \sum_{n=0}^{N/2-1} x(n+N/2)W_N^{nk}$$

$$= \sum_{n=0}^{N/2-1} \left[x(n) + (-1)^{k} x\left(n+\frac{N}{2}\right) \right] W_N^{nk}$$

$$\sum_{n=0}^{N/2-1} \left[x(n) + (-1)^{nk}\, x\left(n + \frac{N}{2}\right) \right] W_N^{nk} \qquad X(2)$$

Let us split $X(k)$ into even and odd numbered samples. For even values of k, the $X(k)$ can bewritten as

$$X(2K) = \sum_{n=0}^{N/2-1} \left[x(n) + (-1)^{2k}\, x\left(n + \frac{N}{2}\right) \right] W_N^{2nk}$$

$$= \sum_{n=0}^{N/2-1} \left[x(n) + x\left(n + \frac{N}{2}\right) \right] W_N^{nk}$$

For odd values of k, the X(k) can be written as

$$X(2K+1) = \sum_{n=0}^{N/2-1}\left[x(n) + (-1)^{2k+1} x\left(n+\frac{N}{2}\right)\right] W_N^{(2k+1)n}$$

$$= \sum_{n=0}^{N/2-1}\left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^{nk} W_{N/2}^{nk}$$

The above equations for $X(2k)$ and $X(2k+1)$ can be recognized as $N/2$-point DFTs. $X(2k)$ is the DFT of the sum of first half and last half of the input sequence, i.e. of $\{x(n) + x(n + N/2)\}$ and $X(2k + 1)$ is the DFT of the product W_N^n with the difference of first half and last half of the input, i.e. $\{x(n) - x(n + N/2)\}W_N^n$.
of

If we define new time domain sequences, $u_1(n)$ and $u_2(n)$ consisting of $N/2$-samples, such that

$$u_1(n) = x(n) + x\left(n + \frac{N}{2}\right); \quad \text{for } n = 0, 1, 2, \ldots, \frac{N}{2} - 1$$

and

$$u_2(n) = \left[x(n) - x\left(n + \frac{N}{2}\right)\right] W_N^n; \quad \text{for } n = 0, 1, 2, \ldots, \frac{N}{2} - 1$$

then the DFTs $U_1(k) = X(2k)$ and $U_2(k) = X(2k + 1)$ can be computed by first forming the sequences $u_1(n)$ and $u_2(n)$, then computing the $N/2$-point DFTs of these two sequences to obtain the even numbered output points and odd numbered output points respectively. The procedure suggested above is illustrated in Figure 7.9 for the case of an 8-point sequence.

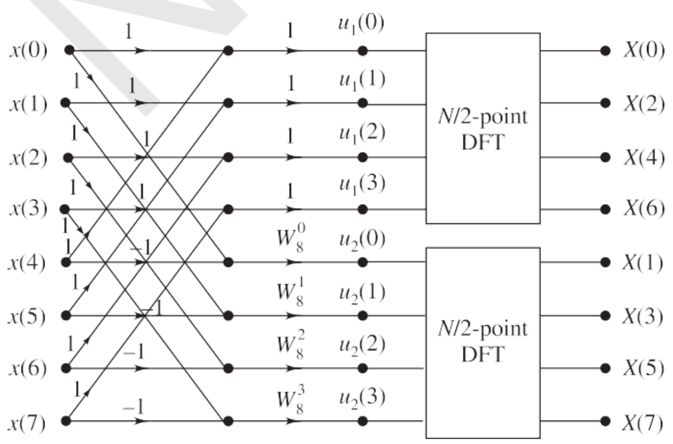


Figure 7.9 FFow graph of the DIF decomposition of an N–point DF† computation into two $N/2$–point DF† computations $N = 8$.

Now each of the N/2-point frequency domain sequences, $U1(k)$ and $U2(k)$ can be decimated into two numbers of N/4-point sequences and four numbers of new N/4-point sequences can be obtained from them.

Let the new sequences be $v11(n)$, $v12(n)$, $v21(n)$, $v22(n)$. On similar lines as discussed above, we can get

$$v_{11}(n) = u_1(n) + u_1(n+2); \quad \text{for } n = 0, 1, 2, \ldots, \frac{N}{4} - 1$$

$$v_{12}(n) = [u_1(n) - u_1(n+2)]W^n_{N/2}; \quad \text{for } n = 0, 1, 2, \ldots, \frac{N}{4} - 1$$

$$v_{21}(n) = u_2(n) + u_2(n+2); \quad \text{for } n = 0, 1, 2, \ldots, \frac{N}{4} - 1$$

$$v_{22}(n) = [u_2(n) - u_2(n+2)]W^n_{N/2}; \quad \text{for } n = 0, 1, 2, \ldots, \frac{N}{4} - 1$$

This process is continued till we get only 2-point sequences. The DFT of those 2-point sequences is the DFT of $x(n)$, i.e. $X(k)$ in bit reversed order.

The third stage of computation for N = 8 is shown in Figure 7.11.

The entire process of decimation involves m stages of decimation where $m = \log 2N$. The computation of the N-point DFT via the DIF FFT algorithm requires (N/2) log2N complex multiplications and (N – 1) log2N complex additions (i.e. total number of computations remains same in both DIF and DIT).

Observing the basic calculations, each stage involves N/2 butterflies of the type shownin Figure 7.12.

The butterfly computation involves the following operations:

(i) In each computation two complex numbers a and b are considered.

(ii) The sum of the two complex numbers is computed which forms a new complexnumber A.

(iii) Subtract the complex number b from a to get the term $(a – b)$. The difference term $(a – b)$ is multiplied with the phase factor or twiddle factorcomplex number B. W_N^n to form a new

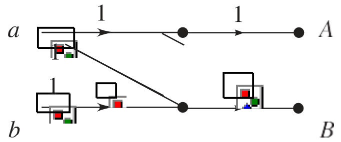


Figure 7.12 Basic butterFy diagram for DIF FI†.

The signal flow graph or butterfly diagram of all the three stages together is shown in Figure7.13.

TKE 8-POINT DFT USIKG RADIX-2 DIF FFT

The DIF computations for an 8-sample sequence are given below in detail.
Let $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$ be the given 8-sample sequence.

First stage of COMPUTATION

In the first stage of computation, two numbers of 4-point sequences $u1(n)$ and $u2(n)$ areobtained from the given 8-point sequence $x(n)$ as shown below.

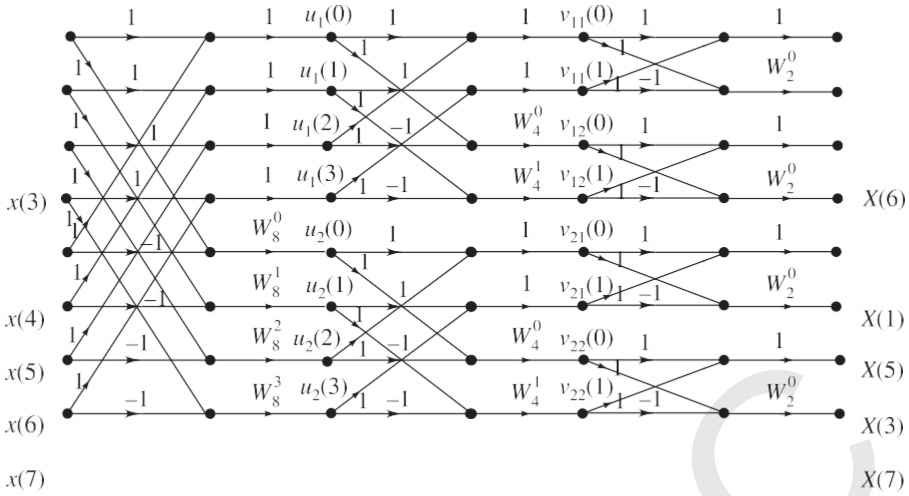


Figure 7.13 SignaF fFow graph or butterfFy diagram for the 8–point radix–2 DIF FF† aFgorithm.

Seconb stage of CONPUTATION

In the second stage of computation, four numbers of 2-point sequences $v11(n)$, $v12(n)$ and $v21(n)$, $v22(n)$ are obtained form the two 4-point sequences $u1(n)$ and $u2(n)$ obtained in stage one

Thirb stage of CONPUTATION

In the third stage of computation, the 2-point DFTs of the 2-point sequences obtained in thesecond stage . The computation of 2-point DFTs is done by the butterfly operatio shown in Figure 7.14(c).

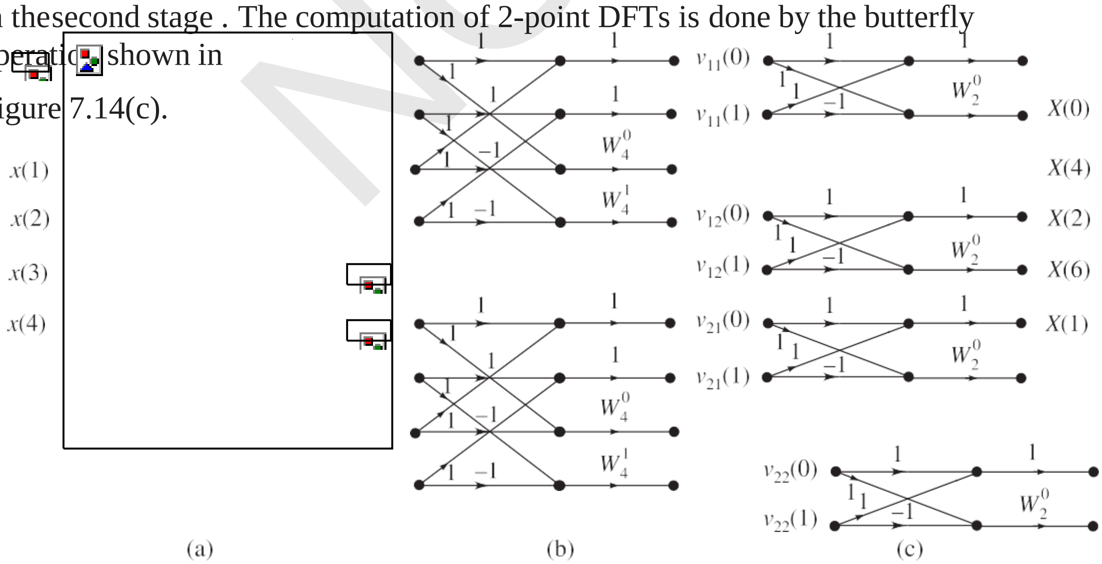


Figure 7.14 (a)–(c) †he first, second and third stages of computation of 8–point DF† by Radix–2 DIF FF†.

CONPARISON of DlT (DECIMATION-IN-TINE) and DlF (DECINATION-IN-FREQUENCY) ALGORITHMS

Difference between DIT and DIF

1. In DIT, the input is bit reversed while the output is in normal order. For DIF, the reverse is true, i.e. the input is in normal order, while the output is bit reversed. However, both DIT and DIF can go from normal to shuffled data

2. or vice versa.
3. Considering the butterfly diagram, in DIT, the complex multiplication takes place before the add subtract operation, while in DIF, the complex multiplication takes place after the add subtract operation.

Similarities

1. Both algorithms require the same number of operations to compute DFT.
2. Both algorithms require bit reversal at some place during computation.

7.6.f Computation of IDFT through FFT

The IDFT of an N-point sequence $\{X(k)\}$; $k = 0, 1, ..., N-1$ is defined as

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}nk} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-nk}$$

Taking the conjugate of the above equation for $x(n)$, we get

$$x^*(n) = \left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-nk}\right]^* = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)\, W_N^{nk}$$

Taking the conjugate of the above equation for $x^*(n)$, we get

$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{N-1} X^*(k)\, W_N^{nk}\right]^*$$

The term inside the square brackets in the above equation for $x(n)$ is same as the DFT computation of a sequence $X^*(k)$ and may be computed using any FFT algorithm. So we can say that the IDFT of $X(k)$ can be obtained by finding the DFT of $X^*(k)$, taking the conjugate of that DFT and dividing by N. Hence, to compute the IDFT of $X(k)$ the following procedure can be followed

1. Take conjugate of $X(k)$, i.e. determine $X^*(k)$.
2. Compute the N-point DFT of $X^*(k)$ using radix-2 FFT.
3. Take conjugate of the output sequence of FFT.
4. Divide the sequence obtained in step-3 by N.

The resultant sequence is $x(n)$.Thus, a single FFT algorithm serves the evaluation of both direct andinverse DFTs.

EXAMPLE 1 Draw the butterfly line diagram for 8-point FFT calculation and briefly explain. Use decimation-in-time algorithm.

Solution: The butterfly line diagram for 8-point DIT FFT algorithm is shown in following Figure

Solution: For 8-point DIT FFT

1. The input sequence $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$,
2. bit reversed order, of input as i.e. as $x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\}$. Since $N = 2^m = 2^3$, the 8-point DFT computation
3. Radix-2 FFT involves 3 stages of computation, each stage involving 4 butterflies. The output

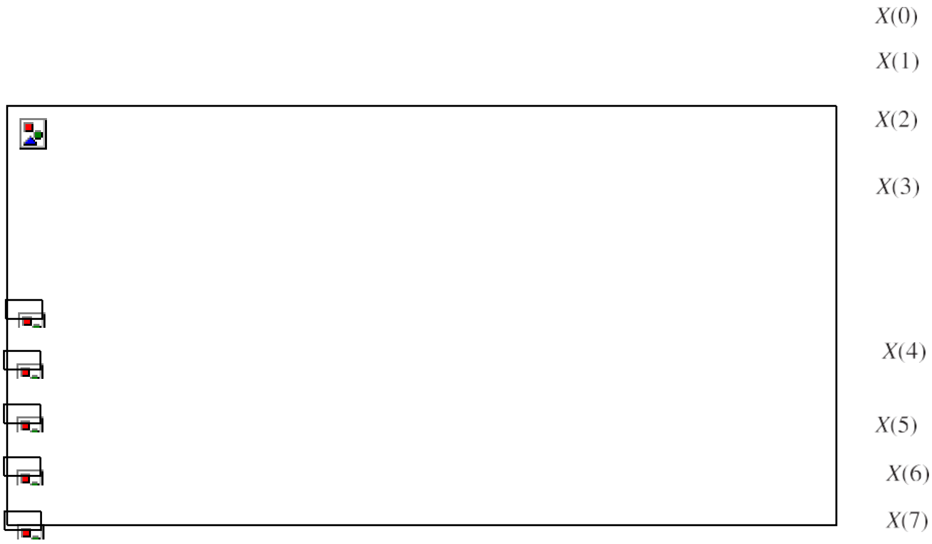
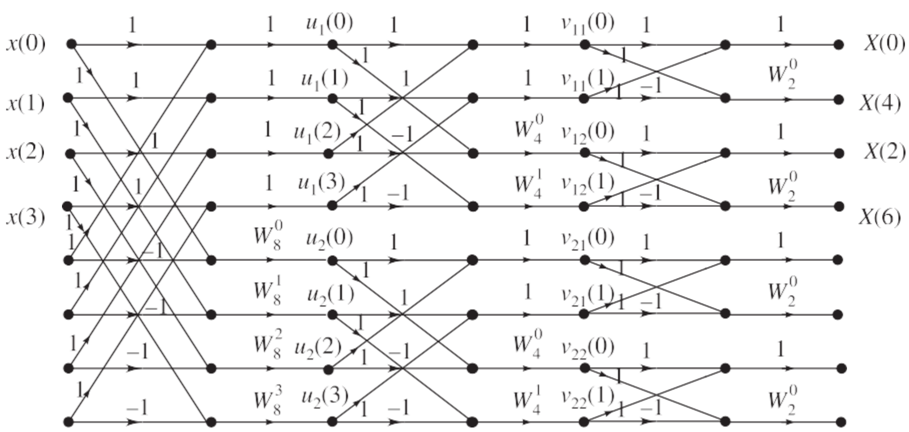


Figure : Butterfly Fine diagram for 8–point DIT FFT algorithm for $N = 8$.

EXAMPLE 2 Implement the decimation-in-frequency FFT algorithm of N-point DFT where $N = 8$. Also explain the steps involved in this algorithm.

Solution: The 8-point radix-2 DIF FFT algorithm

1. It involves 3 stages of computation. The input to the first stage is the input time sequence $x(n)$ in normal order. The output of first stage is the input to the second stage and the output of second stage is the input to the third stage. The output of third stage is the 8- point DFT in bit reversed order.
2. In DIF algorithm, the frequency domain sequence $X(k)$ is decimated.
3. In this algorithm, the N-point time domain sequence is converted to two numbers of $N/2$- point sequences. Then each $N/2$-point sequence is converted to two numbers of $N/4$-point sequences. Thus, we get 4 numbers of $N/4$, i.e. 2-point sequences.
4. Finally, the 2-point DFT of each 2-point sequence is computed. The 2-point DFTs of $N/2$ number of 2-point sequences will give N-samples which is the N-point DFT of the time domain sequence. The implementation of the 8-point radix-2 DIF FFT algorithm is shown in Figure 7.16.



x(5) X(5)

x(6) X(3)

x(7) X(7)

Figure 7.16 Butterfly Fine diagram for 8–point radix–2 DIF FFT algorithm.

EXAMPLE 7.4 What is FFT? Calculate the number of multiplications needed in the calculation of DFT using FFT algorithm with 32-point sequence.

Solution: The FFT, i.e. Fast Fourier transform is a method (or algorithm) for computing the DFT with reduced number of calculations. The computational efficiency is achieved by adopting a divide and conquer approach. This approach is based on the decomposition of an N- point DFT into successively smaller DFTs. This basic approach leads to a family of efficient computational algorithms known as FFT algorithms. Basically there are two FFT algorithms. (i) DIT FFT algorithm and (ii) DIF FFT algorithm. If the length of the sequence $N = 2^m$, 2 indicates the radix and m indicates the number of stages in the computation. In radix-2 FFT, the N-point sequence is decimated into two $N/2$-point sequences, each $N/2$-point sequence is decimated into two $N/4$-point sequences and so on till we get two point sequences. The DFTs of two point sequences are computed and DFTs of two 2-point sequences are combined into DFT of one 4-point sequence, DFTs of two 4-point sequences are combined into DFT of one 8-point sequence and so on till we get the N-point DFT.

The number of multiplications needed in the computation of DFT using FFT algorithm with $N = 32$-point sequence is $= \frac{N}{2} \log_2 N = \frac{32}{2} \log_2 2^5 = 80$.

The number of complex additions $= N \log_2 N = 32 \log_2 32 = 32 \log_2 2^5 = 160$

EXAMPLE 7.5 Explain the inverse FFT algorithm to compute inverse DFT of a 8-point DFT. Draw the flow graph for the same.

Solution: The IDFT of an 8-point sequence $\{X(k), k = 0, 1, 2, ..., 7\}$ is defined as

$$x(n) = \frac{1}{8}\sum_{k=0}^{7} X(k)\, W_8^{-nk},\; n = 0, 1, 2, \ldots, 7$$

Taking the conjugate of the above equation for $x(n)$, we have

$$x^*(n) = \frac{1}{8}\left[\sum_{k=0}^{7} X^*(k) W_8^{nk}\right]$$

Taking the conjugate of the above equation for $x^*(n)$ we have

$$x(n) = \frac{1}{8}\left[\sum_{k=0}^{7} X^*(k) W_8^{nk}\right]^*$$

The term inside the square brackets in the RHS of the above expression for $x(n)$ is the 8-point DFT of $X^*(k)$.

Hence, in order to compute the IDFT of $X(k)$ the following procedure can be followed:

1. Given $X(k)$, take conjugate of $X(k)$ i.e. determine $X^*(k)$.

2 Compute the DFT of $X^*(k)$ using radix-2 DIT or DIF FFT, [This gives $8x^*(n)$]

1. Take conjugate of output sequence of FFT. This gives $8x(n)$.
2. Divide the sequence obtained in step 3 by 8. The resultant sequence is $x(n)$.

The flow graph for computation of N = 8-point IDFT using DIT FFT algorithm is shown in Figure 7.18.

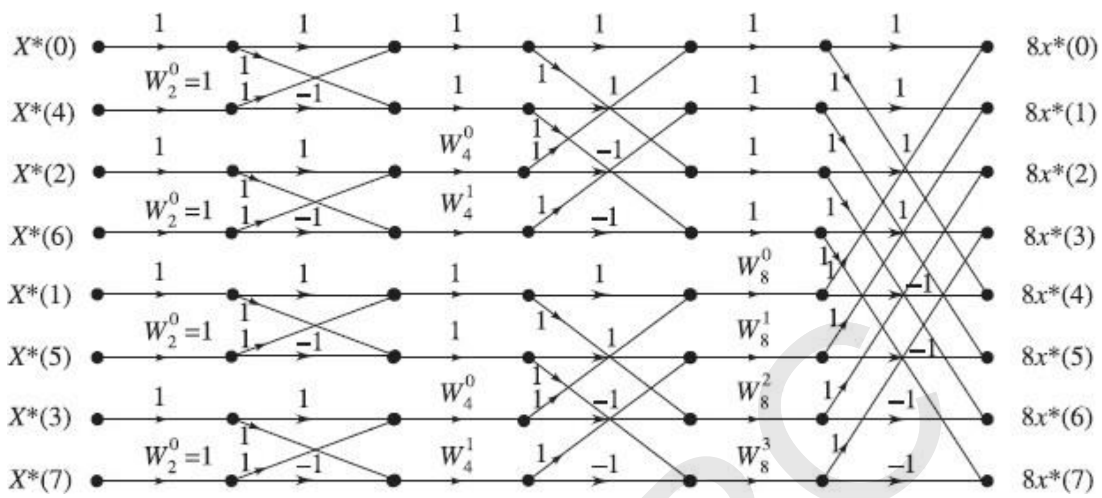


Figure 7.18 Computation of 8–point DF† of $X^*(k)$ by radix–2, DI† FF†.

From Figure 7.18, we get the 8-point DFT of $X^*(k)$ by DIT FFT as

$$8x^*(n) = \{8x^*(0), 8x^*(1), 8x^*(2), 8x^*(3), 8x^*(4), 8x^*(5), 8x^*(6), 8x^*(7)\}$$

$$x(n) = \frac{1}{8}\{8x^*(0), 8x^*(1), 8x^*(2), 8x^*(3), 8x^*(4), 8x^*(5), 8x^*(6), 8x^*(7)\}^*$$

EXAMPLE 7.11 Compute the DFT of the sequence $x(n)$ = {1, 0, 0, 0, 0, 0, 0, 0} (a) directly, (b) by FFT.

Solution: (a) Direct computation of DFT

The given sequence is $x(n)$ = {1, 0, 0, 0, 0, 0, 0, 0}. We have to compute 8-point DFT. So N = 8.

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}nk} = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} = \sum_{n=0}^{7} x(n)\, W_8^{nk}$$

$$= x(0)W_8^0 + x(1)W_8^1 + x(2)W_8^2 + x(3)W_8^3 + x(4)W_8^4 + x(5)W_8^5 + x(6)W_8^6 + x(7)W_8^7$$

$$= (1)\,(1) + (0)\,(W_8^1) + (0)W_8^2 + (0)\,W_8^3 + (0)W_8^4 + (0)W_8^5 + (0)W_8^6 + (0)W_8^7 = 1$$

$X(k) = 1$ for all k

$X(0) = 1,\ X(1) = 1,\ X(2) = 1,\ X(3) = 1,\ X(4) = 1,\ X(5) = 1,\ X(6) = 1,\ X(7) = 1$

$X(k) = \{1, 1, 1, 1, 1, 1, 1, 1\}$

(b) Computation by FFT. Here $N = 8 = 2^3$

The computation of 8-point DFT of $x(n) = \{1, 0, 0, 0, 0, 0, 0, 0\}$ by radix-2 DIT FFT algorithm is shown in Figure 7.31. $x(n)$ in bit reverse order is

$$x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\}$$

$$= \{1, 0, 0, 0, 0, 0, 0, 0\}$$

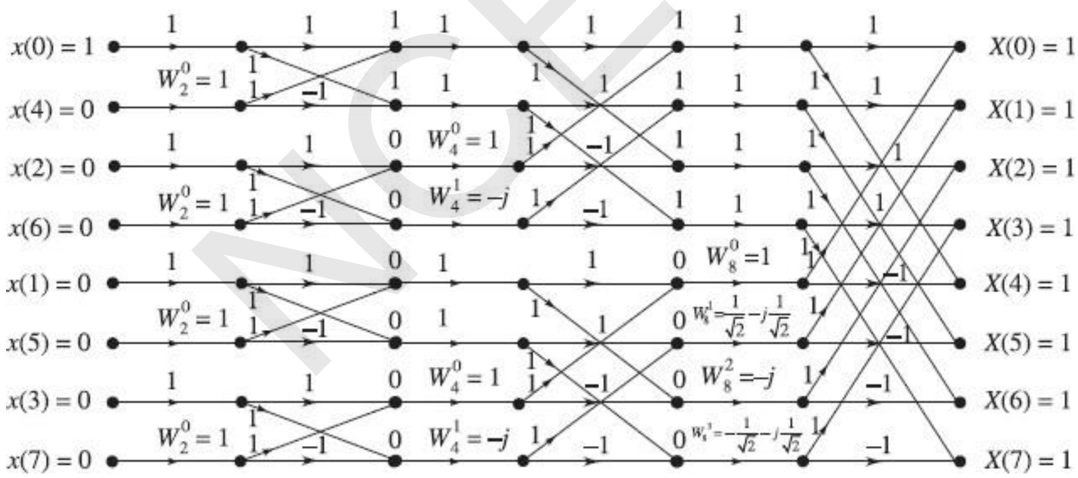
For DIT FFT input is in bit reversed order and output is in normal order.

From Figure 7.31, the 8-point DFT of the given $x(n)$ is $X(k) = \{1, 1, 1, 1, 1, 1, 1, 1\}$

EXAMPLE 7.12 An 8-point sequence is given by $x(n) = \{2, 2, 2, 2, 1, 1, 1, 1\}$. Compute the 8-point DFT of $x(n)$ by

(a) Radix-2 DIT FFT algorithm
(b) Radix-2 DIF FFT algorithm

Also sketch the magnitude and phase spectrum.



Solution: (a) 8-point DFT by Radix-2 DIT FFT algorithm

The given sequence is $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$

$$= \{2, 2, 2, 2, 1, 1, 1, 1\}$$

The given sequence in bit reversed order is

$$x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\}$$

$$= \{2, 1, 2, 1, 2, 1, 2, 1\}$$

For DIT FFT, the input is in bit reversed order and the output is in normal order. The computation of 8-point DFT of $x(n)$, i.e. $X(k)$ by Radix-2 DIT FFT algorithm is shown in Figure 7.32.

From Figure 7.32, we get the 8-point DFT of $x(n)$ as

$$X(k) = \{12,\ 1 - j2.414,\ 0,\ 1 - j0.414,\ 0,\ 1 + j0.414,\ 0,\ 1 + j2.414\}$$

(b) 8-point DFT by radix-2 DIF FFT algorithm
For DIF FFT, the input is in normal order and the output is in bit reversed order. The computation of DFT by radix-2 DIF FFT algorithm is shown in Figure 7.33.

Figure 7.33 Computation of 8–point DF† of $x(n)$ by radix–2 DIF FF† aFgorithm.

From Figure 7.33, we observe that the 8-point DFT in bit reversed order is

$$X_r(k) = \{X(0), X(4), X(2), X(6), X(1), X(5), X(3), X(7)\}$$
$$= \{12,\ 0,\ 0,\ 0,\ 1 - j2.414,\ 1 + j0.414,\ 1 - j0.414,\ 1 + j2.414\}$$

The 8-point DFT in normal order is

$$X(k) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$$
$$= \{12,\ 1 - j2.414,\ 0,\ 1 - j0.414,\ 0,\ 1 + j0.414,\ 0,\ 1 + j2.414\}$$

Magnitude and Phase Spectrum

Each element of the sequence $X(k)$ is a complex number and they are expressed in rectangular coordinates. If they are converted to polar coordinates, then the magnitude and phase of each element can be obtained.

The magnitude spectrum is the plot of the magnitude of each sample of $X(k)$ as a function of k. The phase spectrum is the plot of phase of each sample of $X(k)$ as a function of k. When N-point DFT is performed on a sequence $x(n)$ then the DFT sequence $X(k)$ will have a periodicity of N. Hence, in this example, the magnitude and phase spectrum will have a periodicity of 8 as shown below.

$X(k) = \{12, 1 \; j2.414, 0, 1 \; j0.414, 0, 1 + j0.414, 0, 1 + j2.414\}$

$= \{12 \; 0, 2.61 \; 67, 0 \; 0, 1.08 \; 22, 0 \; 0, 1.08 \; 22, 0 \; 0, 2.61 \; 67\}$

$= \{12 \; 0, 2.61 \; 0.37, 0 \; 0, 1.08 \; 0.12, 0 \; 0, 1.08 \; 0.12, 0 \; 0, 2.61 \; 0.37\}$

$|X(k)| = \{12, 2.61, 0, 1.08, 0, 1.08, 0, 2.61\}$

$\angle X(k) = \{0, \; 0.37, 0, \; 0.12, 0, 0.12, 0, 0.37\}$

The magnitude and phase spectrum are shown in Figures 7.34(a) and (b).

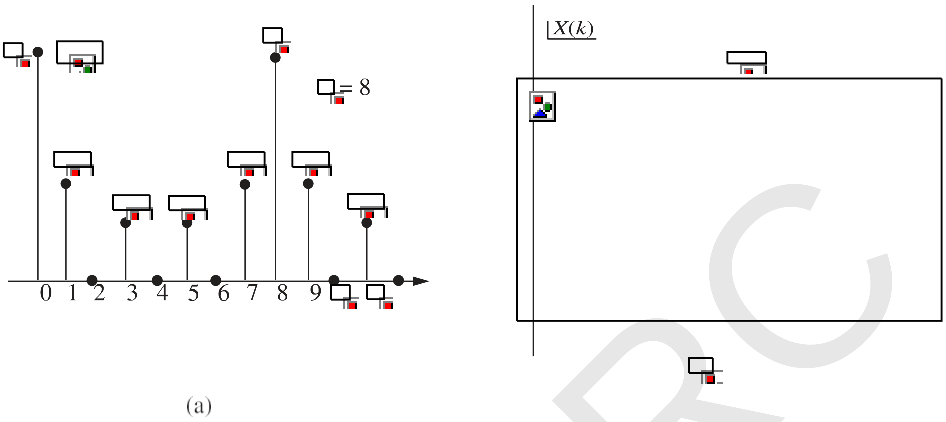


Figure 7.34 (a) Magnitude spectrum, (b) Phase spectrum.

EXAMPLE 7.13 Find the 8-point DFT by radix-2 DIT FFT algorithm.

$x(n) = \{2, 1, 2, 1, 2, 1, 2, 1\}$

Solution: The given sequence is $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$

k

$= \{2, 1, 2, 1, 2, 1, 2, 1\}$

For DIT FFT computation, the input sequence must be in bit reversed order and the outputsequence will be in normal order.

$x(n)$ in bit reverse order is

$x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\}$

$= \{2, 2, 2, 2, 1, 1, 1, 1\}$

The computation of 8-point DFT of $x(n)$ by radix-2 DIT FFT algorithm is shown in Figure 7.35.

From Figure 7.35, we get the 8-point DFT of $x(n)$ as $X(k) = \{12, 0, 0, 0, 4, 0, 0, 0\}$

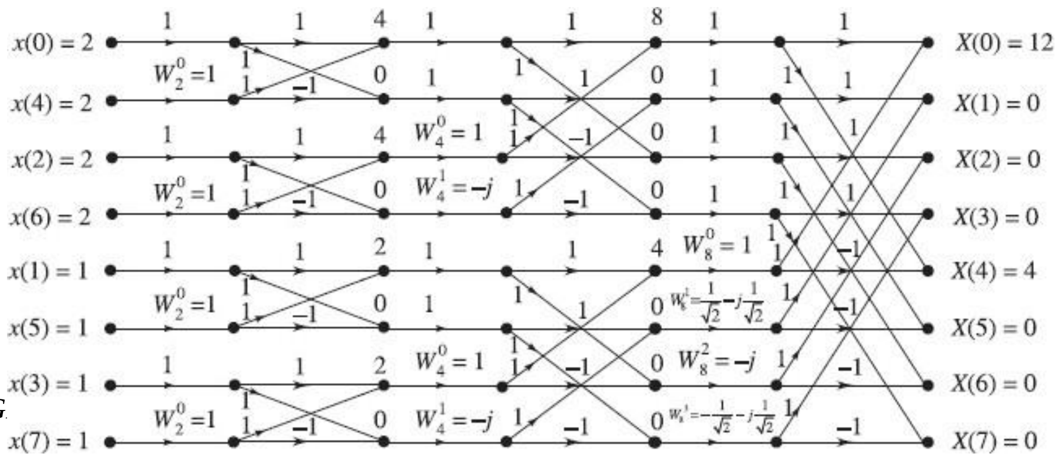


Figure 7.35 Computation of 8–point DF† of $x(n)$ by radix–2, DI† FF†.

EXAMPLE 7.14 Compute the DFT for the sequence $x(n) = \{1, 1, 1, 1, 1, 1, 1, 1\}$.

Solution: The given sequence is $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$

$$= \{1, 1, 1, 1, 1, 1, 1, 1\}$$

The computation of 8-point DFT of $x(n)$, i.e. $X(k)$ by radix-2, DIT FFT algorithm is shownin Figure 7.36.

The given sequence in bit reversed order is

$$x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\}$$

$$= \{1, 1, 1, 1, 1, 1, 1, 1\}$$

For DIT FFT, the input is in bit reversed order and output is in normal order.

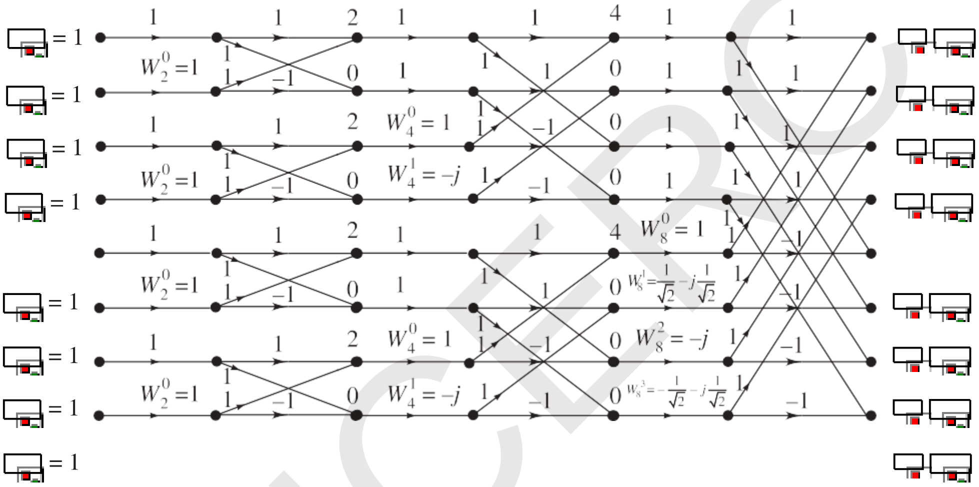


Figure 7.36 Computation of 8–point DF† of $x(n)$ by radix–2, DI† FF†.

From Figure 7.36, we get the 8-point DFT of $x(n)$ as $X(k) = \{8, 0, 0, 0, 0, 0, 0, 0\}$.

EXAMPLE 7.15 Given a sequence $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$, determine $X(k)$ using DIT FFT algorithm.

Solution: The given sequence is $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$

$$= \{1, 2, 3, 4, 4, 3, 2, 1\}$$

The computation of 8-point DFT of $x(n)$, i.e. $X(k)$ by radix-2, DIT FFT algorithm is shown in Figure 7.37. For DIT FFT, the input is in bit reversed order and the output is in normal order.

The given sequence in bit reverse order is

$$x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\} = \{1, 4, 3, 2, 2, 3, 4, 1\}$$

Figure 7.37 Computation of 8–point DF† of $x(n)$ by radix–2, DI† FF†.

From Figure 7.37, we get the 8-point DFT of $x(n)$ as

$X(k) = \{20, 5.828\ j2.414, 0, 0.172\ j0.41\ 4, 0, 0.172 + j0.414, 0, 5.828 + j2.414\}$

EXAMPLE 7.16 Given a sequence $x(n) = \{0, 1, 2, 3, 4, 5, 6, 7\}$, determine $X(k)$ using DIT FFT algorithm.

Solution: The given sequence is $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$

$$= \{0, 1, 2, 3, 4, 5, 6, 7\}$$

The computation of 8-point DFT of $x(n)$, i.e. $X(k)$ by radix-2, DIT FFT algorithm is shown in Figure 7.38. For DIT FFT, the input is in bit reversed order and output is in normal order.

The given sequence in bit reverse order is

$$x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\}$$

$$= \{0, 4, 2, 6, 1, 5, 3, 7\}$$

Figure 7.38 Computation of 8–point DF† of *x*(*n*) by radix–2, DI† FF†.

From Figure 7.38, we get the 8-point DFT of *x*(*n*) as

$X(k)$ = {28, 4 + *j*9.656, 4 + *j*4, 4 + *j*1.656, 4, 4 – *j*1.656, 4 – *j*4, 4 – *j*9.656}

EXAMPLE 7.18 Find the IDFT of the sequence

$$X(k) = \{4, 1 - j2.414, 0, 1 - j0.414, 0, 1 + j0.414, 0, 1 + j2.414\}$$

using DIF algorithm.

Solution: The IDFT *x*(*n*) of the given 8-point sequence *X*(*k*) can be obtained by finding $X^*(k)$, the conjugate of *X*(*k*), finding the 8-point DFT of $X^*(k)$, using DIF algorithm to get

$8x^*(n)$, taking the conjugate of that to get $8x(n)$ and then dividing the result by 8 to get $x(n)$. For DIF algorithm, input $X^*(k)$ must be in normal order. The output will be in bit reversedorder for the given $X(k)$.

$$X^*(k) = \{4, 1 + j2.414, 0, 1 + j0.414, 0, 1 - j0.414, 0, 1 - j2.414\}$$

The DFT of $X^*(k)$ using radix-2, DIF FFT algorithm is computed as shown in Figure 7.42.

Figure 7.42 Computation of 8–point DF† of $X^*(k)$ by radix–2 DIF FF†.

From the DIF FFT algorithm of Figure 7.42, we get

$$8x_r^*(n) = \{8, 0, 8, 0, 8, 0, 8, 0\}$$

$$8x_r(n) = \{8, 0, 8, 0, 8, 0, 8, 0\}^* = \{8, 0, 8, 0, 8, 0, 8, 0\}$$

$$x(n) = \frac{1}{8}\{8, 8, 8, 8, 0, 0, 0, 0\} = \{1, 1, 1, 1, 0, 0, 0, 0\}$$

EXAMPLE 7.19 Compute the IDFT of the sequence

$$X(k) = \{7, -0.707 - j0.707, -j, 0.707 - j0.707, 1, 0.707 + j0.707, j, -0.707 + j0.707\}$$

using DIT algorithm.

Solution: The IDFT $x(n)$ of the given sequence $X(k)$ can be obtained by finding $X^*(k)$, the conjugate of $X(k)$, finding the 8-point DFT of $X^*(k)$ using radix-2 DIT FFT algorithm to get $8x^*(n)$, taking the conjugate of that to get $8x(n)$ and then dividing by 8 to get $x(n)$. For DIT FFT, the input $X^*(k)$ must be in bit reverse order. The output $8x^*(n)$ will be in normal order. For the given $X(k)$.

$$X^*(k) = \{7, -0.707 + j0.707, j, 0.707 + j0.707, 1, 0.707 - j0.707, -j, -0.707 - j0.707\}$$

$X^*(k)$ in bit reverse order is

$$X_r^*(k) = \{7, 1, j, -j, -0.707 + j0.707, 0.707 - j0.707, 0.707 + j0.707, -0.707 - j0.707\}$$

The 8-point DFT of $X^*(k)$ using radix-2, DIT FFT algorithm is computed as shown inFigure 7.43.

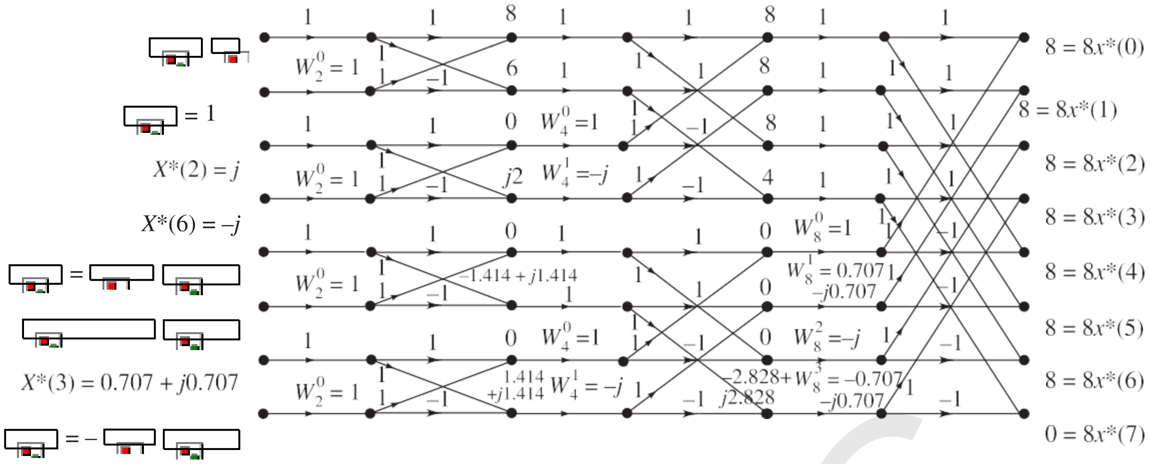


Figure 7.43 Computation of 8–point DF† of $X^*(k)$ by radix–2, DI† FF†.

From the DIT FFT algorithm of Figure 7.43, we have

$$8x^*(n) = \{8, 8, 8, 8, 8, 8, 8, 0\}$$
$$8x(n) = \{8, 8, 8, 8, 8, 8, 8, 0\}$$
$$x(n) = \{1, 1, 1, 1, 1, 1, 1, 0\}$$

EXAMPLE 7.20 Compute the IDFT of the square wave sequence $X(k) = \{12, 0, 0, 0, 4, 0, 0, 0\}$ using DIF algorithm.

Solution: The IDFT $x(n)$ of the given sequence $X(k)$ can be obtained by finding $X^*(k)$, the conjugate of $X(k)$, finding the 8-point DFT of $X^*(k)$ using DIF algorithm to get $8x^*(n)$ taking the conjugate of that to get $8x(n)$ and then dividing the result by 8 to get $x(n)$. For DIF algorithm, the input $X^*(k)$ must be in normal order and the output $8x^*(n)$ will be in bit reversed order.

For the given $X(k)$

$$X^*(k) = \{12, 0, 0, 0, 4, 0, 0, 0\}$$

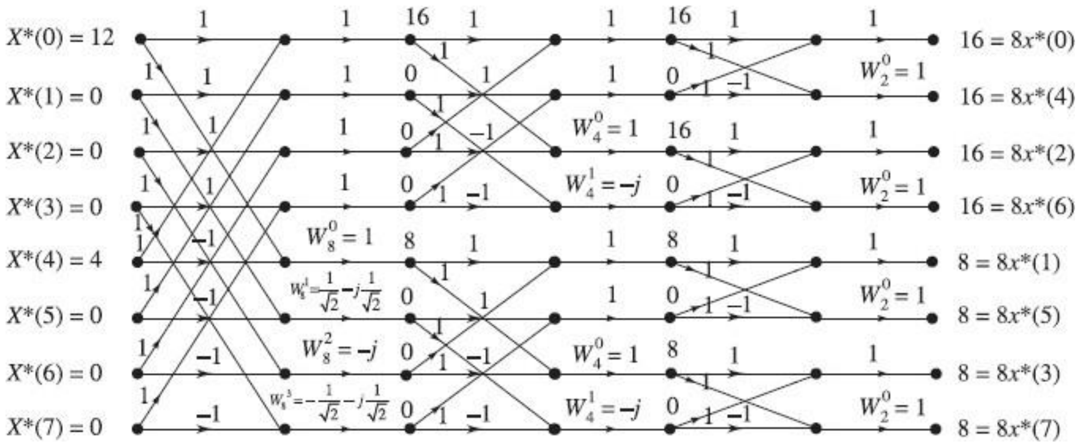
The 8-point DFT of $X^*(k)$ using radix-2, DIF FFT algorithm is computed as shown in Figure7.44.

From Figure 7.44, we have

$$8x_r^*(n) = \{16, 16, 16, 16, 8, 8, 8, 8\}$$

$$8x_r(n) = \{16, 16, 16, 16, 8, 8, 8, 8\}^* = \{16, 16, 16, 16, 8, 8, 8, 8\}$$

$$x(n) = \frac{1}{8}\{16, 8, 16, 8, 16, 8, 16, 8\} = \{2, 1, 2, 1, 2, 1, 2, 1\}$$


X*(0) = 12
X*(1) = 0
X*(2) = 0
X*(3) = 0
X*(4) = 4
X*(5) = 0
X*(6) = 0
X*(7) = 0
16 = 8x*(0)
16 = 8x*(4)
16 = 8x*(2)
16 = 8x*(6)
8 = 8x*(1)
8 = 8x*(5)
8 = 8x*(3)
8 = 8x*(7)

MODULE - 3

3.1 **Design of FIR Filters**

Introduction

↳ In signal processing,

Filter ⇒ Device which removes unwanted features from signal.

↳ FIR ⇒ Finite Impulse Response.

↳ FIR filter ⇒ Impulse Response is finite
(finite no. of samples in impulse response).

↳ O/p FIR filter ⇒ (1) depends only on present and past values.
(OR)
(2) depends only on present or past values.
of i/p samples.

↳ **Applications**

* Major appln. where linear phase is important.

Linear phase :- Phase response of the filter is linear function.

* Linear phase filter ⇒ preserve the shape of the i/p signal → Very important in Communication & medical image as the signal shape is very imp. in these fields.

* Major Applns. ⇒ Data transmission, speech processing, Correlation processing, Interpolation

Characteristics

(i) Impulse response → finite length

(ii) Non-recursive FIR filter → always stable
⇓
(Values are not repeating)

imp char * (iii) Phase distortion that results due to non-linear characteristics of freq. response can be eliminated by FIR filter.

(iv) Possible to implement a recursive FIR filter.
(Current o/p depends on previous o/p)

(v) Effect of start-up transient have small duration

(vi) Quantization noise can be made negligible

Advantages

↳ Stable ⇒ due to non-recursive FIR

↳ Can be realized in both recursive & non-recursive

↳ Exact linear phase

↳ Flexible ⇒ Design procedures → to achieve any magnitude response in FIR filter.

↳ Low sensitive to quantization noise

↳ Efficiently realized in hardware.

Disadvantages

↳ Practical realization of FIR filter → Complex

↳ Requires more filter co-efficients to be stored.

↳ long duration impulse response require large amount of processing.

↳ Costly ⇒ Narrow transition band of FIR filter requires more arithmetic operations & Hardware Components.

Linear Phase FIR filters

Digital filter ⇒ Signals → discrete with r.to. time
$x(n)$

↳ For generating 1KHz signal ⇒ in analog filter → need to design R,C etc. Values

→ in digital filter → need to design filter coefficients

↳ Filters ⇒ frequency selective devices.

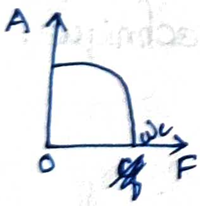
↳ desired frequency response ⇒ $H_d(e^{j\omega})$ ← Frequency domain Fourier transform

↳ nature of filtering action determined by frequency response characteristics $H(e^{j\omega})$

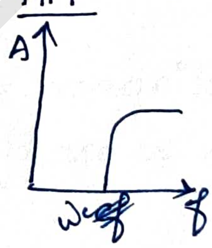
↳ Designing ⇒ freq. response of a filter.

LPF

freq. Res ⇒ low
High freq = 0



HPF



Low freq ⇒ 0
High freq.

Desired frequency response = $H_d(e^{j\omega})$ [Freq. domain]

Signal is in time domain → Convert freq. D to time D by taking IFT of (H_d $e^{j\omega}$)
↳ Inverse fourier transform

Desired impulse response ⇒ $h_d(n)$ Inverse IFT of $H_d(e^{j\omega})$

{Fourier transform ⇒ infinite
freq. varies from $-\infty$ to $+\infty$}

$h_d(n)$ = infinite duration discrete signal ⇒ Values of n $-\infty$ to ∞

$h_d(n)$ range ⇒ $-\infty$ to ∞

- We can't process infinite values in computer. → Required finite values to process
- Designing a digital filter means to find $H(z)$

  Transfer fn. $H(z) =$ z transform of $h_d(n)$.
- Since $h_d(n)$ is indefinite duration sequence → transfer fn $H(z)$ will have infinite terms → which cannot be realised by digital system.
- Therefore finite no. of samples of $h_d(n)$ are taken (selected) → to find the $H(z)$.
- Filter designed by using finite samples of impulse response are called FIR (Finite Impulse Response) Filter.
- infinite to finite ⇒ 2 techniques
  - Window technique
  - Freq. sampling technique.

- Consider discrete time signal $x(n)$

  [Eg:- signal $x(n)$ → have different frequency components need to pass only LP frequencies.

  along with $x(n)$ multiply LPF

  Time domain (convolution) ⇒ frequency multiplication
- Here considering Freq. domain along with signal $x(n)$, LPF passing ⇒ multiplying signal $x(n)$ with corresponding freq. response

LTI ⇒ Linea Time-invariant

Consider discrete time signal $x(n)$
→ its fourier transform ⇒ $X(e^{j\omega})$

↳ Let this signal passed through LTI system with freq. response

H → filter notation

filter response representation (std).

$$H(e^{j\omega}) = \begin{cases} C\,e^{-j\alpha\omega} & \omega_1 \leq \omega \leq \omega_2 \\ 0 & \text{otherwise} \end{cases} \quad \text{—(1)}$$

signal → Phase, Magnitude

where C & α are +ve Constants

$$H(e^{j\omega}) = |H(e^{j\omega})| \angle H(e^{j\omega})$$
$$= C\,e^{-j\alpha\omega} \quad \text{—(2)}$$

where, $|H(e^{j\omega})| = C$ = Magnitude

$\angle H(e^{j\omega}) = -\alpha\omega$ = phase (linear)

The above eqn shows that the magnitude of frequency response is Constant and its phase is a linear function of frequency.

∴ If the phase fn. of freq. response of a filter is linear fn. of a freq., then the filter is called linear phase filter.

* Linear phase filter ⇒ less distortion in phase
* can retrieve the information ⇒ Correctly.

Phase delay and Group delay.

$\angle H(e^{j\omega}) = -\alpha\omega \Rightarrow \theta(\omega)$

~~α ⇒ Constant phase delay.~~

$\therefore\ \theta(\omega) = -\alpha\omega$ —(3)

$$\boxed{\text{Phase delay } \tau_p = \frac{-\theta(\omega)}{\omega}} \quad \text{—(4)}$$

phase / freq

Sub. eqn (3) in (4) $\quad \tau_p = -\left(\frac{-\alpha\omega}{\omega}\right) = +\alpha$ —(5)

Group delay, $\boxed{\tau_g = -\frac{d\theta(\omega)}{d\omega}}$ —(6)

Sub. eqn(3) in eqn(6)

$$\tau_g = \frac{-d(-\alpha\omega)}{d\omega} = \alpha \quad —(7)$$

$$\boxed{\begin{aligned}\tau_p &= \alpha \\ \tau_g &= \alpha\end{aligned}}$$

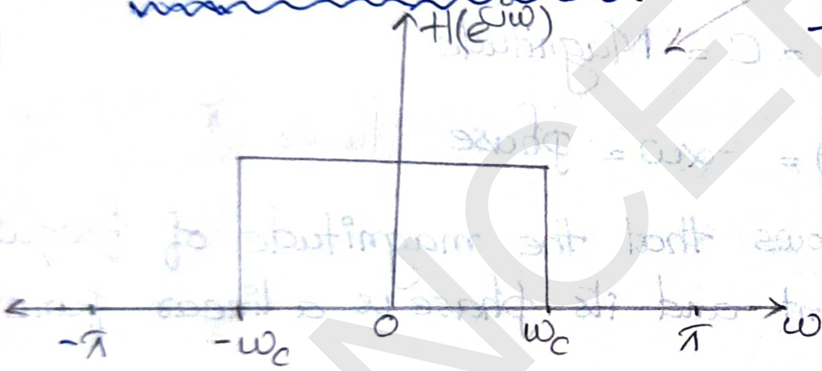
∴ for a linear-phase filter the delay is constant.

$\alpha \Rightarrow$ Constant phase delay.

$\tau_p = \tau_g = \alpha \Rightarrow \alpha$ is independent of freq.

Symmetric: 0 to 2π ⇒ splitting to $-\pi$ to π

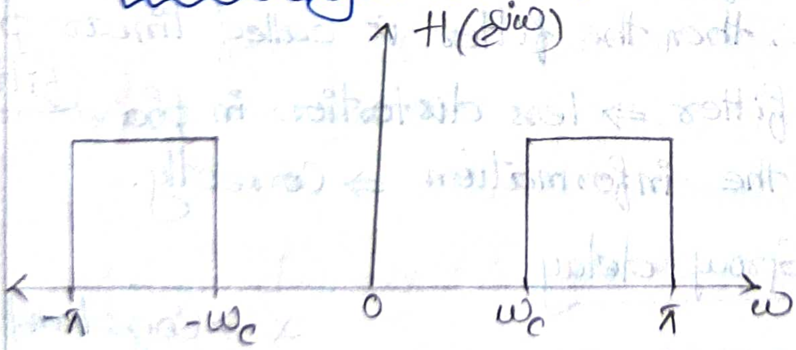
(i) Ideal Low Pass Filter.



$$H_d(\omega) = \begin{cases} Ce^{-j\alpha\omega}, & -\omega_c < \omega < \omega_c \\ 0, & -\pi < \omega < -\omega_c \\ 0, & \omega_c < \omega < \pi \end{cases}$$

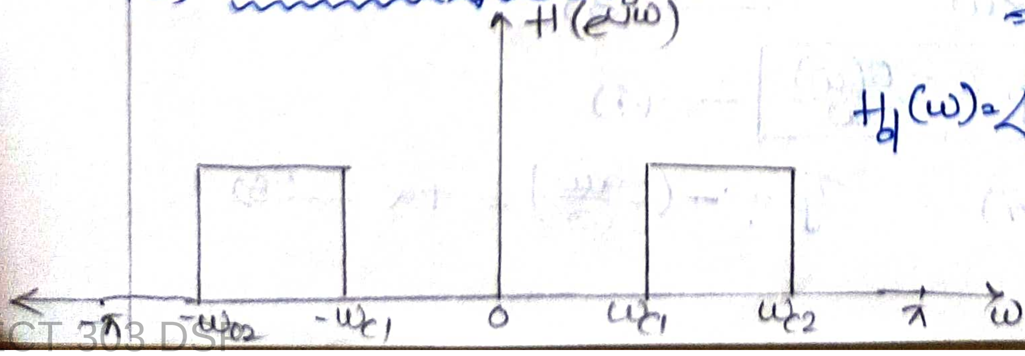
$\omega_c \Rightarrow$ Cut-off freq.

(2) Ideal High pass filter.



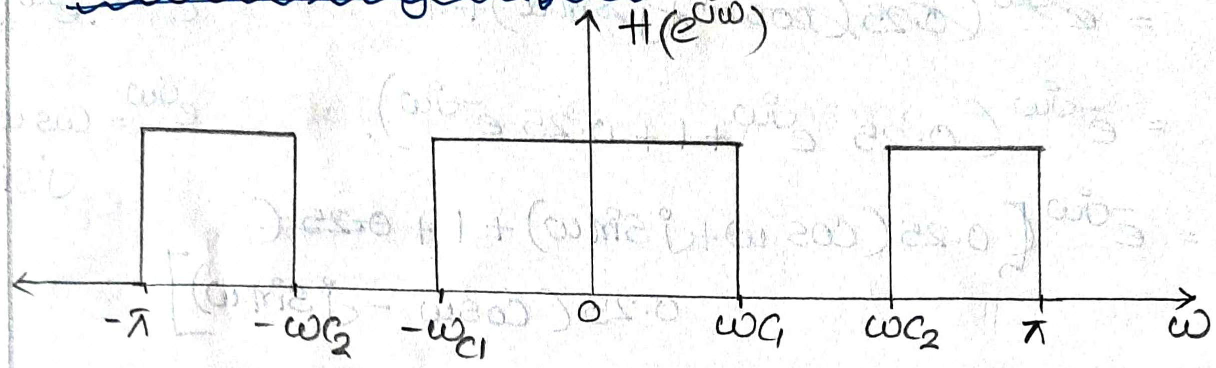
$$H_d(\omega) = \begin{cases} Ce^{-j\alpha\omega}, & \omega_c < \omega < \pi \\ Ce^{-j\alpha\omega}, & -\pi < \omega < -\omega_c \\ 0, & -\omega_c < \omega < \omega_c \end{cases}$$

(3) Ideal Bandpass filter.



$$H_d(\omega) = \begin{cases} Ce^{-j\alpha\omega}, & -\omega_{c2} < \omega < -\omega_{c1} \\ 0, & -\omega_{c1} < \omega < \omega_{c1} \\ Ce^{-j\alpha\omega}, & \omega_{c1} < \omega < \omega_{c2} \\ 0, & \omega_{c2} < \omega < \pi \\ 0, & -\pi < \omega < -\omega_{c2} \end{cases}$$

(4) Ideal Band Reject filter



$$H_d(\omega) = \begin{cases} Ce^{-j\alpha\omega} & ; -\pi < \omega < -\omega c_2 \\ Ce^{-j\alpha\omega} & ; -\omega c_1 < \omega < \omega c_1 \\ Ce^{-j\alpha\omega} & ; \omega c_2 < \omega < \pi \\ 0 & ; -\omega c_2 < \omega < -\omega c_1 \\ 0 & ; \omega c_1 < \omega < \omega c_2 \end{cases}$$

Pbm Determine the freq. response of FIR filter defined by $y(n) = 0.25\, x(n) + x(n-1) + 0.25\, x(n-2)$. Calculate the phase delay and group delay.? (difference eqn.) (→ coefficients)

Phase delay, $\tau_p = \frac{-\theta(\omega)}{\omega}$ — (1)

eqn should be in $Ce^{-j\alpha\omega}$

Group delay $= \tau_g = -\frac{d\theta(\omega)}{d\omega}$ — (2)

$\theta(\omega) = -\alpha\omega$

$y(n) = 0.25\, x(n) + x(n-1) + 0.25\, x(n-2)$ — (3) (shifting property in FT)

Taking fourier transform on both sides.

$$Y(e^{j\omega}) = 0.25\, X(e^{j\omega}) + e^{-j\omega} X(e^{j\omega}) + 0.25\, e^{-j2\omega} X(e^{j\omega})$$ — (4)

Freq. Response, $H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$ — (5)

∴ Divide eqn(4) by $X(e^{j\omega})$

$$H(e^{j\omega}) = 0.25 + e^{-j\omega} + 0.25\, e^{-j2\omega}$$ — (6)

$= e^{-j\omega}(0.25(\cos\omega + j\sin\omega) + 1$ $\quad e^{-j\omega} \cdot e^{j\omega} = 1$

$= e^{-j\omega}(0.25\, e^{j\omega} + 1 + 0.25\, e^{-j\omega})$. $\quad e^{j\omega} = \cos\omega + j\sin\omega$

$= e^{-j\omega}[0.25(\cos\omega + j\sin\omega) + 1 + 0.25(\cos\omega - j\sin\omega)]$

$= e^{-j\omega}[0.25\cos\omega + 0.25 j\sin\omega + 1 + 0.25\cos\omega - 0.25 j\sin\omega]$

$= \underbrace{e^{-j\omega}}_{e^{-j\alpha\omega}}[\underbrace{0.5\cos\omega + 1}_{C}] \Rightarrow C\, e^{-j\alpha\omega}$ $\quad \theta(\omega) = -\omega$

phase delay, $\tau_p = \frac{-\theta(\omega)}{\omega} = \frac{-(-\omega)}{\omega} = 1$

Group delay, $\tau_g = -\frac{d\theta(\omega)}{d\omega} = -\frac{d}{d\omega}(-\omega) = 1$

Linear Phase FIR Filter Design

↳ Let $h(n)$ be an impulse response of a system, then its Fourier Transform can be written as

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} \quad \text{---(1)}$$

FT = $h(n) \times e^{-j\omega n}$ frequency domain

↳ Since $H(e^{j\omega})$ is a complex for linear phase FIR filter, $H(e^{+j\omega})$ can be written as

– Finite length
– Complex quantity
Mag & Phase.

$$H(e^{j\omega}) = \pm |H(e^{j\omega})|\, e^{-j\alpha\omega} \quad \text{---(2)}$$

Equating (1) & (2) $\quad e^{-j\theta} = \cos\theta - j\sin\theta$.

$$\sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} = \pm |H(e^{j\omega})|\, e^{-j\alpha\omega}$$

$$\sum_{n=0}^{N-1} h(n)(\cos\omega n - j\sin\omega n) = \pm |H(e^{j\omega})|(\cos\alpha\omega - j\sin\alpha\omega)$$

Rearrange the above eqn.

$$\sum_{n=0}^{N-1} h(n) \cos \omega n = \pm |H(e^{j\omega})| \cos \alpha\omega \quad \text{——— (3)}$$

$$\sum_{n=0}^{N-1} h(n) \sin \omega n = \pm |H(e^{j\omega})| \sin \alpha\omega \quad \text{——— (4)}$$

Divide eqn (4) by (3).

$$\frac{\sum_{n=0}^{N-1} h(n) \sin \omega n}{\sum_{n=0}^{N-1} h(n) \cos \omega n} = \frac{\pm |H(e^{j\omega})| \sin \alpha\omega}{\pm |H(e^{j\omega})| \cos \alpha\omega}$$

$$\sum_{n=0}^{N-1} h(n) \sin \omega n \cos \alpha\omega = \sum_{n=0}^{N-1} h(n) \cos \omega n \cdot \sin \alpha\omega .$$

$$\sum_{n=0}^{N-1} h(n) \left[\sin \omega n \cos \alpha\omega - \cos \omega n \sin \alpha\omega \right] = 0 .$$

$$0 = \sum_{n=0}^{N-1} h(n) \left(\sin \alpha\omega \cos \omega n - \cos \alpha\omega \sin \omega n \right)$$

$\sin A \cos B - \cos A \sin B = \sin(A-B)$

$$\sum_{n=0}^{N-1} h(n) (\sin \alpha\omega - \omega n) = 0$$

$$\sum_{n=0}^{N-1} h(n) \left[\sin (\alpha - n)\omega \right] = 0 \quad \text{——— (5)}$$

Solution to above eqn. exists when ——— (6)

$*$ $\left\{ \alpha = \frac{N-1}{2} \text{ and } h(n) = h(N-1-n) \text{ for } 0 \le n \le N-1 \right\}$ By heart Condition for Symmetric FIR filter.

$\alpha = \frac{N-1}{2} \Rightarrow$ Centre of Symmetry.

↳ From this condition it can be observed that the impulse response is symmetric about centre of sequence.

For eg:- $N=9$

For $N=$ odd, $\alpha = \frac{N-1}{2} = \frac{9-1}{2} = \frac{8}{2} = 4$, (Centre of Symmetry)

(i) Take $n=3$,

$h(n) = h(N-1-n)$

$h(3) = h(9-1-3) = h(5)$

$h(3) = h(5)$.

(ii) if $n=2$

$h(2) = h(9-1-2) = h(6)$

$h(2) = h(6)$.

shows the symmetry.

(a)

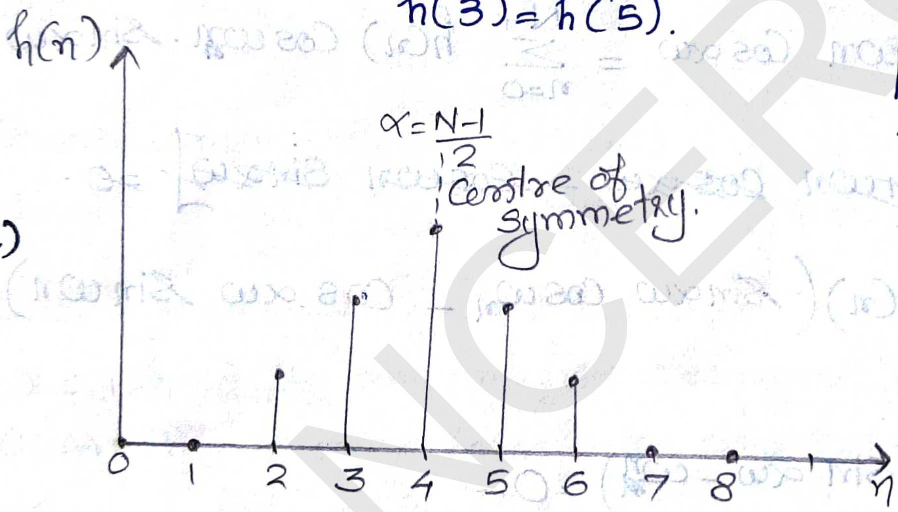


Fig :- Impulse response sequence of symmetric sequences for
(a) $N=$ odd
(b) $N=$ even.

For $N=$ even

$\alpha = \frac{N-1}{2}$ eg: $N=6$ $\alpha = \frac{6-1}{2} = \frac{5}{2} = 2.5$

& $n=2$, $h(2) = h(6-1-2)$

$h(2) = h(3)$

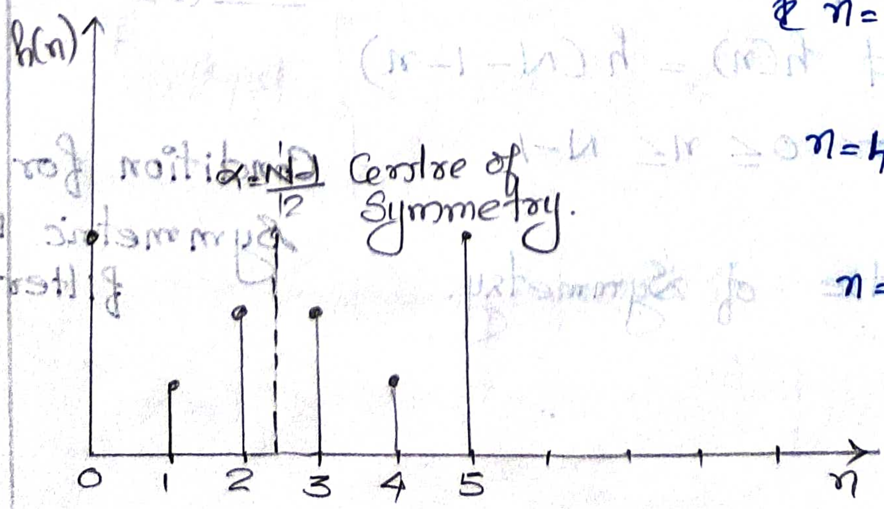
$n=4$, $h(4) = h(6-1-4)$

$h(4) = h(1)$

$n=5$ $h(5) = h(6-1-5) = h(0)$

$h(5) = h(0)$.

(b)



Symmetry ⇒ L.H.S = R.H.S (Same) in the centre of symmetry.

Antisymmetric impulse response

- The definition of linear filter $\theta(\omega) = -\alpha\omega$ requires filter to have both group delay and phase delay constants.
- Suppose only constant group delay is required & phase delay is not required.

then $\theta(\omega) = \beta - \alpha\omega$ —— (7)

then eqn (2) becomes ---- phase delay is not constant

$$H(e^{j\omega}) = \pm |H(e^{j\omega})| e^{j(\beta-\alpha\omega)} \quad \text{—— (8)}$$

Equating (1) & (8)

$$\sum_{n=0}^{N-1} h(n) e^{-j\omega n} = \pm |H(e^{j\omega})| e^{j(\beta-\alpha\omega)}$$

$$\sum_{n=0}^{N-1} h(n)(\cos\omega n - j\sin\omega n) = \pm |H(e^{j\omega})| (\cos(\beta-\alpha\omega) + j\sin(\beta-\alpha\omega))$$

Equate Cos & sin terms.

$$\sum_{n=0}^{N-1} h(n)\cos\omega n = \pm |H(e^{j\omega})| \cos(\beta-\alpha\omega) \quad \text{—— (9)}$$

$$-\sum_{n=0}^{N-1} h(n)\sin\omega n = \pm |H(e^{j\omega})| \sin(\beta-\alpha\omega) \quad \text{—— (10)}$$

Divide eqn (10) by (9)

$\sin A \cos B - \cos A \sin B = \sin(A-B)$

$$\frac{-\sum_{n=0}^{N-1} h(n)\sin\omega n}{\sum_{n=0}^{N-1} h(n)\cos\omega n} = \frac{\sin(\beta-\alpha\omega)}{\cos(\beta-\alpha\omega)} \quad \text{—— (11)}$$

$$-\sum_{n=0}^{N-1} h(n)\sin\omega n \cdot \cos(\beta-\alpha\omega) = \sum_{n=0}^{N-1} h(n)\cos\omega n \sin(\beta-\alpha\omega)$$

~~Rearrange the eqn.~~

~~$\sum_{n=0}^{N-1} h(n)[\sin(\beta-\alpha\omega)\cdot\cos\omega n]$~~

Solving the above eqn (11)

$$\sum_{n=0}^{N-1} h(n) \sin(\beta - \alpha\omega - \omega n) = 0.$$

$$\sum_{n=0}^{N-1} h(n) \sin[\beta - (\alpha - n)\omega] = 0.$$

$\sin(\frac{\pi}{2} - \theta) = \cos\theta$.

↳ if $\beta = \frac{\pi}{2}$, then

$$\sum_{n=0}^{N-1} h(n) \sin\left[\frac{\pi}{2} - (\alpha - n)\omega\right] = 0$$

$$\Rightarrow \sum_{n=0}^{N-1} h(n) \cos(\alpha - n)\omega = 0 \quad \text{———} (12).$$

↳ Solution to above eqn. (12) exists when.

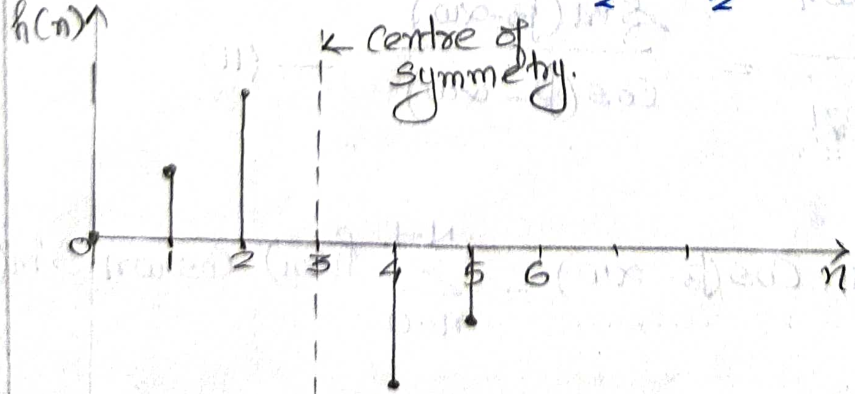
$$h(n) = -h(N-1-n)$$
$$\& \quad \alpha = \frac{N-1}{2}$$

<u>condition</u> ——— (13)

∴ FIR filter have constant group delay (τ_g) and phase delay (τ_p) is not constant.

↳ Eg:- When impulse response is antisymmetrical about $\alpha = \frac{N-1}{2}$

<u>For N = odd</u>, $N = 7$ $\alpha = \frac{7-1}{2} = \frac{6}{2} = 3.$



$h(n) = -h(N-1-n)$

$h(0) = -h(7-1-0)$

$h(0) = -h(6)$

$h(1) = -h(7-1-1) = -h(5)$

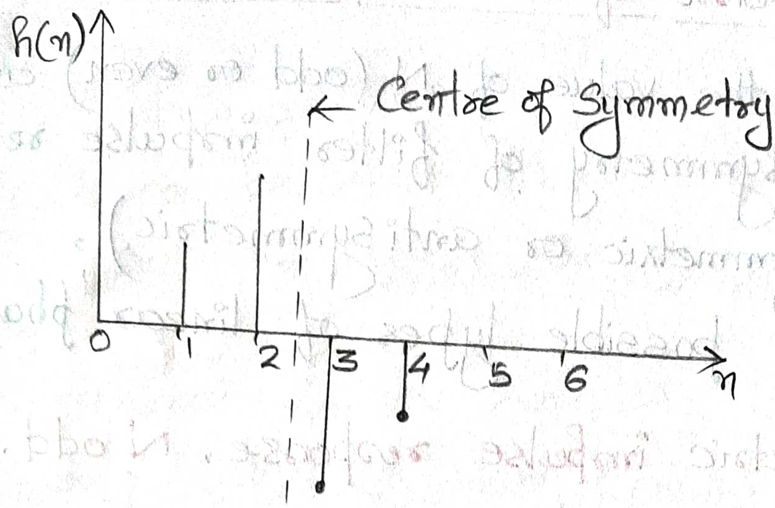
$h(2) = -h(7-1-2) = h(-4)$

satisfies the antisymmetry.

For N = even, N = 6, $\alpha = \frac{N-1}{2} = \frac{6-1}{2} = \frac{5}{2} = 2.5$.

$h(n) = -h(N-1-n)$.

$h(2) = -h(6-1-2)$
$= -h(3)$.



III ↳ When the centre of symmetry $\alpha = 0$, we can have symmetric or antisymmetric sequence for N = odd & N = even number.

When $\alpha = 0$,

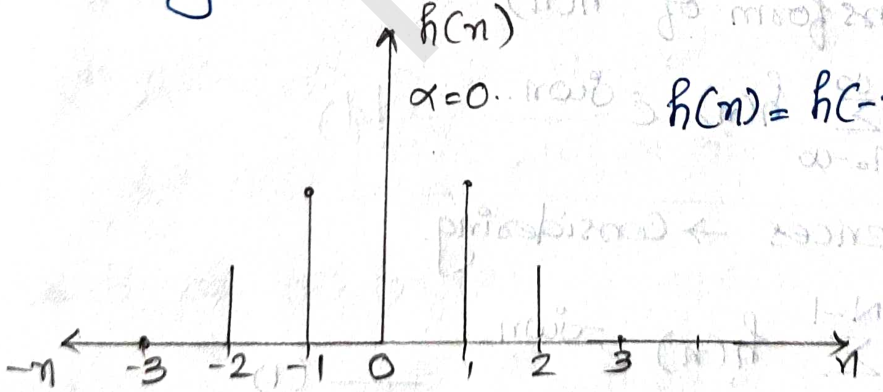
$\alpha = \frac{N-1}{2} \Rightarrow \frac{N-1}{2} = 0 \quad \therefore$ ~~N=0~~. $N-1 = 0$.

$\therefore$ Symmetric, $h(n) = h(N-1-n) \Rightarrow h(n) = h(-n)$

Antisymmetric, $h(n) = -h(N-1-n) \Rightarrow h(n) = -h(-n)$

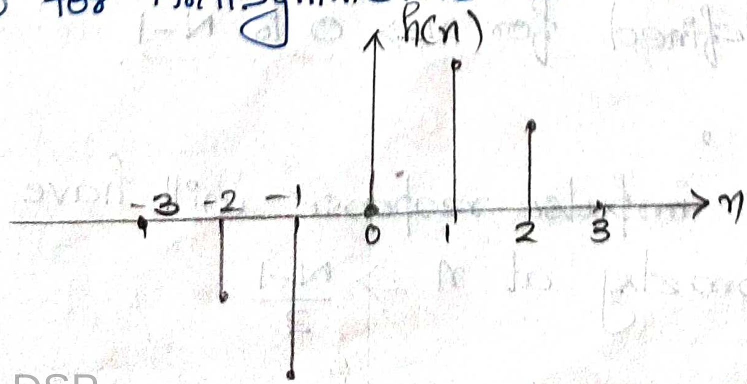
(a) For Symmetric, at $\alpha = 0$.

$h(n) = h(-n)$



(b) For Antisymmetric at $\alpha = 0$.

$h(n) = -h(-n)$.



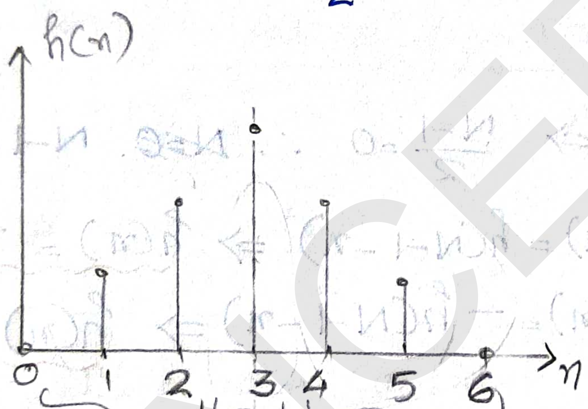
Frequency Response of Linear phase FIR filters.

- Depending on the value of N (odd or even) and the type of symmetry of filter impulse response sequence (symmetric or antisymmetric), there are 6 possible types of linear phase.

Case I : Symmetric impulse response, N odd.

- Symmetric impulse response
- N = odd with centre of ~~freq~~ symmetry at $\frac{N-1}{2}$

Eg:- $N = 7 \quad \alpha = \frac{N-1}{2} \quad \alpha = \frac{7-1}{2} = 6/2 = 3.$



Gn = h(n)
Need to find $H(e^{j\omega})$.
freq. rep.

- Fourier transform of h(n)

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n}. \quad (std)$$

- finite sequences → Considering.

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} \quad \text{—— (1)}$$

- Since h(n) is defined for n ⇒ 0 to N-1 interval.
- When n ⇒ odd no.,

The symmetric impulse response will have centre of symmetry at n ⇒ $\frac{N-1}{2}$

↳ eqn (1) can be ~~split~~ written ~~like~~ as, N = 7

$$H(e^{j\omega}) = \sum_{n=0}^{6} h(n) e^{-j\omega n}$$

↳ This can be split like

$$H(e^{j\omega}) = \sum_{n=0}^{2} h(n) e^{-j\omega n} + h(3) e^{-j3\omega} + \sum_{n=4}^{6} h(n) e^{-j\omega n} \quad \text{— (2)}$$

↳ In general, for N samples.

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) e^{-j\omega n}.$$

↳ Let $h(n) = h(N-1-n)$. — (3)

Let $m = N-1-n$

$n = N-1-m$ — (6)

↳ When $n = \frac{N+1}{2} \Rightarrow \frac{N+1}{2} = N-1-m$

$m = N-1-\left(\frac{N+1}{2}\right) \Rightarrow m = \frac{2N-2-N-1}{2} = \frac{N-3}{2}$

$\boxed{m = \frac{N-3}{2}, \text{ when } n = \frac{N+1}{2}}$ — (7).

↳ when $n = N-1$

from (6) $\Rightarrow N-1 = N-1-m$

$\therefore \boxed{m = 0, \text{ when } n = N-1}$ — (8)

$\sum_{\frac{N-3}{2}}^{m=0} \Rightarrow \sum_{m=0}^{N-3/2}$ lower to higher

↳ Sub eqn (6), (7), (8) on eqn (3).

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega(N-1)/2} + \sum_{m=0}^{(N-3)/2} h(N-1-m) e^{-j\omega(N-1-m)}$$

↳ Put $m = n$

$$H(e^{j\omega}) = \sum_{n=0}^{(N-3)/2} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{(N-3)/2} h(N-1-n) e^{-j\omega(N-1-n)} \quad \text{— (9).}$$

↳ For a symmetrical impulse response

$$h(n) = h(N-1-n).$$

↳ Sub. $h(n) = h(N-1-n)$ in eqn (9)

$$2N-2-2n-N+1 \over 2$$
$$\frac{N-1}{2} - \frac{2n}{2}$$
$$\frac{N-1}{2} - n$$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega(N-1-n)}$$

$$= e^{-j\omega\frac{(N-1)}{2}} \left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{(N-3)/2} h(n) \left(e^{-j\omega n} \cdot e^{j\omega\left(\frac{N-1}{2}\right)} + e^{-j\omega(N-1-n)} \cdot e^{j\omega\left(\frac{N-1}{2}\right)} \right) \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{(N-3)/2} h(n) \left(e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(N-1-n-\frac{N-1}{2}\right)} \right) \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n) \left(e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{2N-2-2n-N+1}{2}\right)} \right) \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{(N-3)/2} h(n) \left(e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right) \right]$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\therefore 2\cos\theta = e^{j\theta} + e^{-j\theta}.$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{(N-3)/2} h(n)\; 2\cos\left(\frac{N-1}{2} - n\right)\omega \right]$$

Re-arranging.

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{(N-3)/2} 2h(n) \cos\omega\left(\frac{N-1}{2} - n\right) \right] \quad \text{—(10)}$$

↳ Let $k = \frac{N-1}{2} - n$

$$n = \frac{N-1}{2} - k$$

Limits $n = 0 \Rightarrow 0 = \frac{N-1}{2} - k$

$$\therefore \boxed{k = \frac{N-1}{2}}$$

$$n = \frac{N-3}{2} \Rightarrow \frac{N-3}{2} = \frac{N-1}{2} - k$$

$$k = \frac{N-1}{2} - \frac{N-3}{2} = 1 \quad \therefore \boxed{k = 1}$$

Sub the values of n and in eqn (10).

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + \sum_{k=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-k\right)\cos\omega k\right]$$

Put $k=n$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right)\cos\omega n\right]$$

The freq. response

$$H(e^{j\omega}) = \underbrace{e^{-j\omega\left(\frac{N-1}{2}\right)}}_{\text{phase fn}}\underbrace{\left[h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{(N-1)/2} 2h\left(\frac{N-1}{2}-n\right)\cos\omega n\right]}_{\text{Amplitude fn}}$$

Let $H(e^{j\omega}) = A(\omega)e^{j\theta(\omega)}$.

$A(\omega) =$ Amplitude fn.

$e^{j\theta(\omega)} =$ Phase fn.

↳ Here

$A(\omega) =$ Amplitude fn

$$= h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{(N-1)/2} 2h\left(\frac{N-1}{2}-n\right)\cos\omega n$$

$\theta(\omega) =$ phase fn.

$$= -\omega\left(\frac{N-1}{2}\right) \Rightarrow -\omega\alpha \quad \left(\alpha = \frac{N-1}{2}\right)$$

↳ Magnitude fn.

$$|H(e^{j\omega})| = |A(\omega)| = \left|h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\left(\frac{N-1}{2}\right)} 2h\left(\frac{N-1}{2}-n\right)\cos\omega n\right|$$

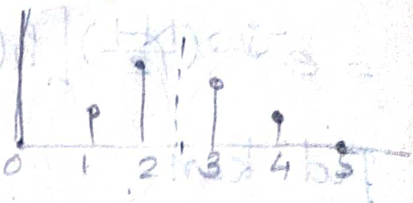
Case II : Symmetric impulse response for, N even.

F.T. of $h(n)$, $H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n}$ —(1)

eg: $N = 6$.

$\alpha = \frac{N-1}{2}$ $\quad h(n) = h(N-1-n)$.

$\alpha = 2.5$



$$H(e^{j\omega}) = \sum_{n=0}^{5} h(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{2} h(n) e^{-j\omega n} + \sum_{n=3}^{5} h(n) e^{-j\omega n}.$$

↳ General for N samples

$$H(e^{j\omega}) = \sum_{n=0}^{(N-2)/2} h(n) e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n) e^{-j\omega n} \quad —(2).$$

For symmetric impulse response with even no. of samples (N = even), Centre of symmetry lies between

$n = \frac{N-2}{2} = \frac{N}{2} - 1$ and $n = \frac{N}{2}$

Eg:- $N = 6$ Centre of Symmetry $\frac{N-1}{2} = 2.5$.

2.5 lies b/w 2 and 3
($2 \to \frac{N}{2}-1$, $3 \to \frac{N}{2}$)

↳ Let $m = N-1-n$

$\therefore n = N-1-m$ —(3)

limits when (i) $n = \frac{N}{2} \Rightarrow \frac{N}{2} = N-1-m$.

$m = N-1-\frac{N}{2}$

$= \frac{2N-2-N}{2}$

$= \frac{N-2}{2} = \frac{N}{2}-1$ —(4)

When (ii) $n = N-1 \Rightarrow N-1 = N-1-m$

$m = N-1-N+1 = 0$

$m = 0$. —(5).

sub. values in eqn(2)

$$= \sum_{n=0}^{(N/2)-1} h(n) e^{-j\omega n} + \sum_{m=0}^{\frac{N}{2}-1} h(N-1-m) e^{-j\omega(N-1-m)}$$

for making same variable

put $m = n$

$$e^{-j\omega\left[N-1-n-\left(\frac{N-1}{2}\right)\right]}$$

$$2N-2-2n-N+1$$

$$(N-1-2n)/2$$

$$= \sum_{n=0}^{(N/2)-1} h(n)\, e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(N-1-n)\, e^{-j\omega(N-1-n)} \quad \text{—(6)}$$

↳ by symmetric condition $h(n) = h(N-1-n)$.

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega(N-1-n)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega \frac{N-1}{2}} \left[e^{-j\omega n} e^{j\omega\left(\frac{N-1}{2}\right)} + e^{-j\omega(N-1-n)} e^{j\omega\left(\frac{N-1}{2}\right)} \right]$$

Add exponential terms.

$$= \sum_{n=0}^{\left(\frac{N}{2}-1\right)} h(n)\, e^{-j\omega \frac{N-1}{2}} \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(N-1-n-\left(\frac{N-1}{2}\right)\right)} \right]$$

$$= \sum_{n=0}^{\left(\frac{N}{2}-1\right)} h(n)\, e^{-j\omega \frac{N-1}{2}} \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right]$$

$$* \ \frac{e^{j\theta} + e^{-j\omega}}{2} = \cos\theta \Rightarrow 2\cos\theta = e^{j\theta} + e^{-j\theta}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega\left(\frac{N-1}{2}\right)} \left[2\cos \omega\left(\frac{N-1}{2} - n\right) \right]$$

Rearranging.

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N}{2}-1} 2h(n) \cos\left(\omega\left(\frac{N}{2} - n - \frac{1}{2}\right)\right) \quad \text{—(7)}$$

Let $k = \frac{N}{2} - n \Rightarrow n = \frac{N}{2} - k$ —

limits, when $n = 0 \Rightarrow k = \frac{N}{2}$ —

when $n = \frac{N}{2} - 1$ $\quad \frac{N}{2} - 1 = \frac{N}{2} - k \quad \therefore k = 1$.

Sub n & limit values in eqn (7).

$$= \sum_{k=1}^{N/2} 2h\left(\frac{N}{2} - k\right) e^{-j\omega\left(\frac{N-1}{2}\right)} \cos\left(\omega\left(k - \frac{1}{2}\right)\right)$$

put $k = n$

$$= \sum_{n=1}^{N/2} 2h\left(\frac{N}{2}-n\right) e^{-j\omega\left(\frac{N-1}{2}\right)} \cos\left(\omega\left(n-\frac{1}{2}\right)\right)$$

Freq. Response

$$H(e^{j\omega}) = \sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2}-n\right) \cos\left(\omega\left(n-\frac{1}{2}\right)\right) e^{-j\omega\left(\frac{N-1}{2}\right)}.$$

$$H(e^{j\omega}) = A(\omega)\, e^{j\theta(\omega)}$$

$$A(\omega) = \text{Amplitude fn} = \sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2}-n\right) \cos \omega\left(n-\frac{1}{2}\right)$$

$$\theta(\omega) = \text{Phase fn} = -\omega\frac{(N-1)}{2} = -\omega\alpha \quad \text{where } \alpha = \frac{N-1}{2}$$

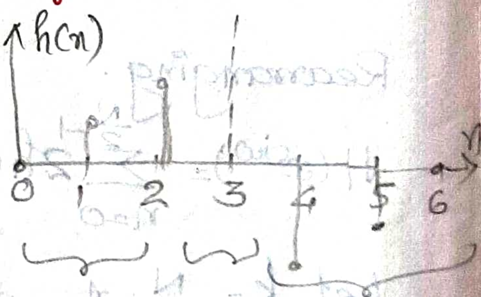
Magnitude fn,

$$|H(e^{j\omega})| = \left|\sum_{n=1}^{N/2} 2h\left(\frac{N}{2}-n\right)\cos\left(\omega\left(n-\frac{1}{2}\right)\right)\right|$$

Case III :- Antisymmetric impulse response, N odd.

FT of $h(n)$

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} \quad \text{--- (1).}$$



N = odd.

cond. $\alpha = \frac{N-1}{2}$ and $h(n) = -h(N-1-n)$

Eg: $N = 7$ $\quad \alpha = \frac{7-1}{2} = 3$

$$\text{eqn(1)} \Rightarrow H(e^{j\omega}) = \sum_{n=0}^{6} h(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{2} h(n)\, e^{-j\omega n} + h(3)\, e^{-j\omega 3} + \sum_{n=4}^{6} h(n)\, e^{-j\omega n} \quad \text{--- (2)}$$

In general

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)\, e^{-j\omega n} \quad \text{--- (3)}$$

$h(n) = -h(N-1-n)$.

Let $m = N-1-n$

$n = N-1-m$ --- (4)

Limits, $n = \frac{N+1}{2} \Rightarrow N-1-m = \frac{N+1}{2}$

$2N-2-2m = N+1$

$2N-2-N-1 = 2m$

$m = \frac{N-3}{2}$ --- (5)

$n = N-1 \Rightarrow N-1 = N-1-m$

$m = N-1-N+1$

$m = 0$. --- (6)

∴ eqn (3) becomes.

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{(N-3)/2} h(N-1-m)\, e^{-j\omega(N-1-m)}.$$

↳ Put $m = n$.

$$\Rightarrow \sum_{n=0}^{\left(\frac{N-3}{2}\right)} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\left(\frac{N-3}{2}\right)} h(N-1-n)\, e^{-j\omega(N-1-n)} \quad \text{--- (7)}$$

$h(n) = -h(N-1-n)$.

$$\therefore \Rightarrow \sum_{n=0}^{\left(\frac{N-3}{2}\right)} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} - \sum_{n=0}^{\left(\frac{N-3}{2}\right)} h(n)\, e^{-j\omega(N-1-n)}$$

$$\Rightarrow e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right)+\sum_{n=0}^{\frac{N-3}{2}} h(n)\left(e^{-j\omega n}\cdot e^{j\omega\left(\frac{N-1}{2}\right)} - e^{-j\omega(N-1-n)}\cdot e^{j\omega\left(\frac{N-1}{2}\right)}\right)\right.$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right)+\sum_{n=0}^{\frac{N-3}{2}} h(n)\left(e^{+j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left(N-1-n-\frac{N-1}{2}\right)}\right)\right.$$

$$(2N-2-2n-N+1)/2$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right)+\sum_{n=0}^{\frac{N-3}{2}} h(n)\left(e^{j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right)\right)$$

$$\left\{\frac{e^{j\theta}-e^{-j\theta}}{2j} = \sin\theta \qquad 2j\sin\theta = e^{j\theta}-e^{-j\theta}.\right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} j\left[h\left(\frac{N-1}{2}\right)+\sum_{n=0}^{\frac{N-3}{2}} h(n)\, 2\sin\left(\frac{N-1}{2}-n\right)\omega\right]$$

Rearrange. $e^{j\pi/2}$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} j\left[h\left(\frac{N-1}{2}\right)+\sum_{n=0}^{\left(\frac{N-3}{2}\right)} 2h(n)\sin\omega\left(\frac{N-1}{2}-n\right)\right] \quad -(8)$$

Let $k = \frac{N-1}{2}-n$

$n = \frac{N-1}{2}-k$

Limits

$n=0, \quad \frac{N-1}{2}-k=0$

$k = \frac{N-1}{2}$ ✓

$n = \frac{N-3}{2}, \quad \frac{N-3}{2} = k = \frac{N-3}{2}$

$k = \frac{N-1}{2}-\left(\frac{N-3}{2}\right)$

$= \frac{N-1-N+3}{2} = 2/2 = 1$

$k=1$ ✓

eqn (8) becomes

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} j\left[h\left(\frac{N-1}{2}\right)+\sum_{k=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-k\right)\sin\omega(k)\right]$$

(with $e^{j\pi/2}$ written above j; "2 h(n) sin ω" struck out) put $k = n$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} e^{j\pi/2} \left[h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right) \sin \omega n \right]$$

$$= e^{j\left(\frac{\pi}{2} - \omega\left(\frac{N-1}{2}\right)\right)} \left[h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right) \sin \omega n \right].$$

—(9).

$A(\omega)$ = Amplitude fn $= \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right) \sin \omega n$

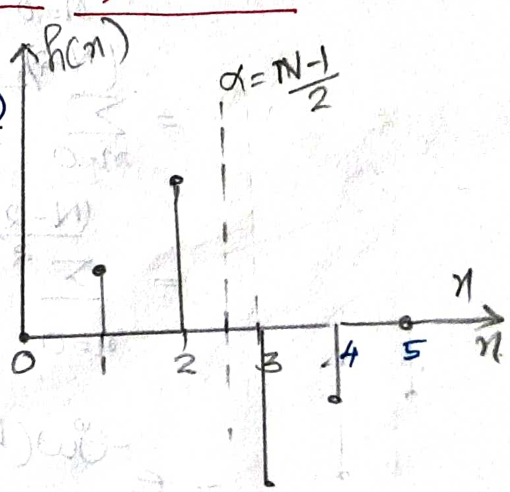
$\theta(\omega)$ = phase fn $= \left(\frac{\pi}{2} - \omega\left(\frac{N-1}{2}\right)\right) = \beta - \alpha\omega.$

Magnitude fn.

$$|H(e^{j\omega})| = |A(\omega)| = \left| \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right) \sin \omega n \right| \quad \text{—(10).}$$

Case IV :- Asymmetric Impulse Response, N even

FT of $h(n) \Rightarrow H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n}$ —(1)



N = even No.

Eg: N = 6 $\quad \alpha = \frac{N-1}{2} = \frac{6-1}{2} = \frac{5}{2} = 2.5.$

∴ eqn (1) becomes

$$H(e^{j\omega}) = \sum_{n=0}^{5} h(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{2} h(n) e^{-j\omega n} + \sum_{n=3}^{5} h(n) e^{-j\omega n}$$

↳ For General samples,

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-2}{2}} h(n) e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n) e^{-j\omega n} \quad \text{——(2)}$$

↳ centre of symmetry ⇒ lies b/w 2 and 3 (from fig)

∴ lies b/w $\frac{N-2}{2}$ and $\frac{N}{2}$

Let $m = N-1-n$

$n = N-1-m$

Limits (i) $n = \frac{N}{2}$, then $N-1-m = \frac{N}{2}$

$m = N-1-\frac{N}{2} = \frac{2N-2-N}{2}$

$= \frac{N-2}{2}$

$\boxed{m = \frac{N}{2}-1}$

(ii) $n = N-1$, then $N-1-m = N-1$

$\boxed{m=0}$

$$\therefore H(e^{j\omega}) = \sum_{n=0}^{\frac{N-2}{2}} h(n) e^{-j\omega n} + \sum_{m=0}^{\frac{N-2}{2}} h(N-1-m) e^{-j\omega(N-1-m)}.$$

put $m = n$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-2}{2}} h(n) e^{-j\omega n} + \sum_{n=0}^{\frac{N-2}{2}} h(N-1-n) e^{-j\omega(N-1-n)}$$

$h(n) = -h(N-1-n)$

$$= \sum_{n=0}^{(N-2)/2} h(n) e^{-j\omega n} - \sum_{n=0}^{\left(\frac{N-2}{2}\right)} h(n) e^{-j\omega(N-1-n)}$$

$$= \sum_{n=0}^{\frac{(N-2)}{2}} h(n) \left[e^{-j\omega n} - e^{-j\omega(N-1-n)}\right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{(N-2/2)} h(n) \left[e^{-j\omega n} \cdot e^{j\omega\left(\frac{N-1}{2}\right)} - e^{-j\omega(N-1-n)} e^{j\omega\left(\frac{N-1}{2}\right)}\right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\left(\frac{N-2}{2}\right)} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left(N-1-n-\left(\frac{N-1}{2}\right)\right)}\right]$$

$$\left\{ \frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta \qquad e^{j\theta} - e^{-j\theta} = 2j\sin\theta \right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\left(\frac{N-2}{2}\right)} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{(N-2)/2} h(n)\, 2j \sin\omega\left(\frac{N-1}{2}-n\right)$$

$\left(\frac{N}{2}-n-\frac{1}{2}\right)$

Ass

$H(e^{j\omega}) = e^{-j\omega(\frac{N-1}{2})} e^{j\pi/2} \left[\sum_{n=1}^{N/2} 2h(\frac{N}{2}-n) \right]$

Let $k = \frac{N}{2} - n$

$\therefore n = \frac{N}{2} - k$

Limits

(i) $n = 0 \Rightarrow \frac{N}{2} - k = 0$

$k = \frac{N}{2}$

(ii) $n = \frac{N-2}{2} \Rightarrow \frac{N-2}{2} = \frac{N}{2} - k$

$k = \frac{N}{2} - (\frac{N-2}{2}) = \frac{N-N+2}{2} = 1$

$$= e^{-j\omega(\frac{N-1}{2})} e^{j\pi/2} \left[\sum_{k=1}^{(N/2)} 2h(\frac{N}{2}-k) \sin\omega(k-\frac{1}{2}) \right]$$

put $k = n$

$$= e^{-j\omega(\frac{N-1}{2})} e^{j\pi/2} \left[\sum_{n=1}^{(N/2)} 2h(\frac{N}{2}-n) \sin\omega(n-\frac{1}{2}) \right].$$

$$H(e^{j\omega}) = e^{j(\frac{\pi}{2} - \omega(\frac{N-1}{2}))} \left[\sum_{n=1}^{N/2} 2h(\frac{N}{2}-n) \sin\omega(n-\frac{1}{2}) \right]$$

Amplitude fn

$$A(\omega) = \left[\sum_{n=1}^{N/2} 2h(\frac{N}{2}-n) \sin\omega(n-\frac{1}{2}) \right].$$

phase fn.

$$\theta(\omega) = \frac{\pi}{2} - \omega(\frac{N-1}{2}) = \beta - \alpha\omega$$

Case v

Frequency response of linear phase FIR filter when impulse response is symmetric and N is odd with centre of symmetry at $n=0$.

$\alpha = 0$

$$H(e^{j\omega}) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n$$

$$A(\omega) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n$$

$$\theta(\omega) = 0.$$

$$|H(e^{j\omega})| = |A(\omega)| = \left| h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n \right|$$

Case vi

Frequency response of linear phase FIR filter when impulse response is antisymmetric and N is odd with centre of antisymmetry at $n=0$.

$$H(e^{j\omega}) = \left[\sum_{n=1}^{\left(\frac{N-1}{2}\right)} 2h(n)\sin\omega n\right] e^{-j\pi/2}$$

$$A(\omega) = \sum_{n=1}^{\left(\frac{N-1}{2}\right)} 2h(n)\sin\omega n$$

$$\theta(\omega) = -\frac{\pi}{2}$$

$$|H(e^{j\omega})| = \left| \sum_{n=1}^{\left(\frac{N-1}{2}\right)} 2h(n)\sin\omega n \right|$$

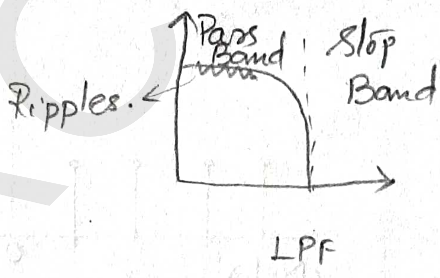
FIR FILTER DESIGN TECHNIQUES.

1. Window method ✓
2. Frequency sampling method ✓
3. Optimum equiripple method. ×

infinite duration impulse response ⇒ finite duration ⇓ Using Window.

1. Window method.

- Windows are finite duration sequence used to modify the impulse response of FIR Filter → to reduce the ripples in the pass band and stop band.
- Also to achieve the desired transition from pass band to stop band.

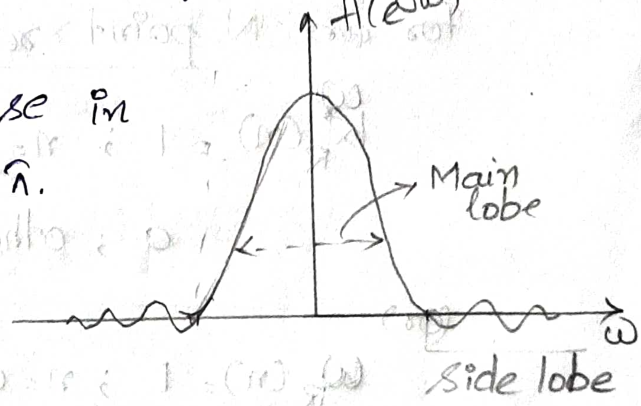


- The desirable characteristics of freq. response of a window fn. are

(i) The width of the main lobe should be small and it should contain as much of total energy as possible.

(ii) The side lobe should decrease in energy rapidly as $\omega \to \pi$.

*(0 to 2π or $-\pi$ to π)



Design procedure for FIR filter using Windows.

(i) Choose the desired freq. response of the filter $H_d(e^{j\omega})$

(ii) Take the inverse FT of $H_d(e^{j\omega})$ to obtain $h_d(n)$.

(iii) Choose a window sequence $W(n)$ and multiply it with $h_d(n)$ to convert the infinite duration impulse response to finite duration impulse response.

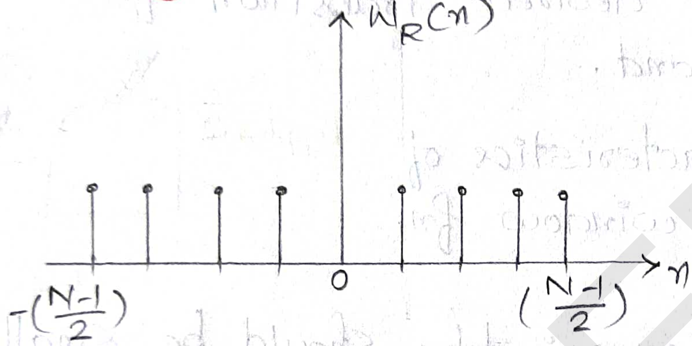
$$h(n) = h_d(n) \times W(n)$$

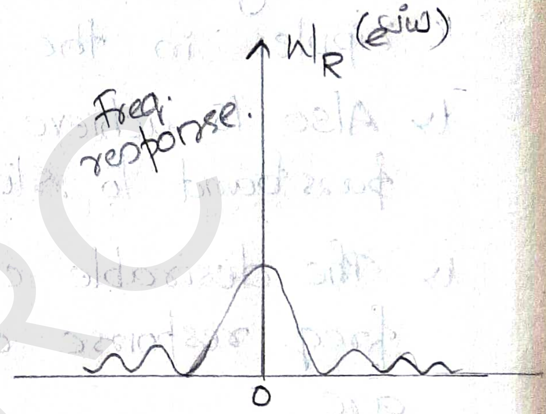
(iv) The transfer fn. $H(z)$ of the filter is obtained by taking Z-transform of $h(n)$.

Types of Windows

1. Rectangular Window ✓
2. Triangular Window / Bartlett Window ×
3. Hanning Window ✓
4. Hamming Window ✓
5. Raised Cosine Window ×
6. Blackman window ×

Rectangular Window





For an N point rectangular window.

$$w_R(n) = \begin{cases} 1 \; ; & n = -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right) \\ 0 \; ; & \text{otherwise} \end{cases} \Bigg\} \text{ if } N = \text{odd.}$$

(or)

$$w_R(n) = \begin{cases} 1 \; ; & n = 0 \text{ to } N-1 \\ 0 \; ; & \text{otherwise} \end{cases} \Bigg\} \text{ if } N = \text{odd or even.}$$

Frequency response of rectangular window.

$$w_R(n) = \begin{cases} 1 & \text{for } n = -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right) \\ 0 & \text{for otherwise.} \end{cases}$$

$W_R(e^{j\omega}) = \text{F.T. of } w_R(n)$

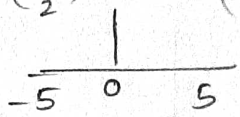
$$W_R(e^{j\omega}) = \frac{\sin \frac{\omega N}{2}}{\sin \frac{\omega}{2}}$$

Proof

$$W_R(e^{j\omega}) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N+1}{2}} 1 \cdot e^{-j\omega n}$$

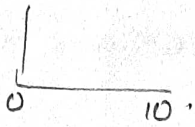
$-\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$

Eg.



-5 to 5

=> 0 to 10 · shift



$$= \sum_{n=0}^{N-1} e^{-j\omega\left(n - \frac{N-1}{2}\right)}$$

shifting

$\frac{N-1}{2} + \frac{N-1}{2}$

$N-1+N-2$

$\frac{2N-2}{2} = N-1$

$$= \sum_{n=0}^{N-1} e^{-j\omega n} \cdot e^{+j\omega\left(\frac{N-1}{2}\right)} = e^{j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{N-1} e^{-j\omega n}$$

finite geometric series sum formula.

$$\sum_{n=0}^{N-1} c^n = \frac{1-c^N}{1-c}$$

$$\therefore = e^{j\omega\left(\frac{N-1}{2}\right)} \left(\frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \right)$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)} \frac{e^{j\omega\frac{N}{2}} \cdot e^{-j\omega N/2} - e^{-j\omega\frac{N}{2}} \cdot e^{-j\omega\frac{N}{2}}}{e^{j\omega/2} \cdot e^{-j\omega/2} - e^{-j\omega/2} \cdot e^{-j\omega/2}}$$

$e^{j\omega\frac{N}{2}} \cdot e^{-j\omega/2}$

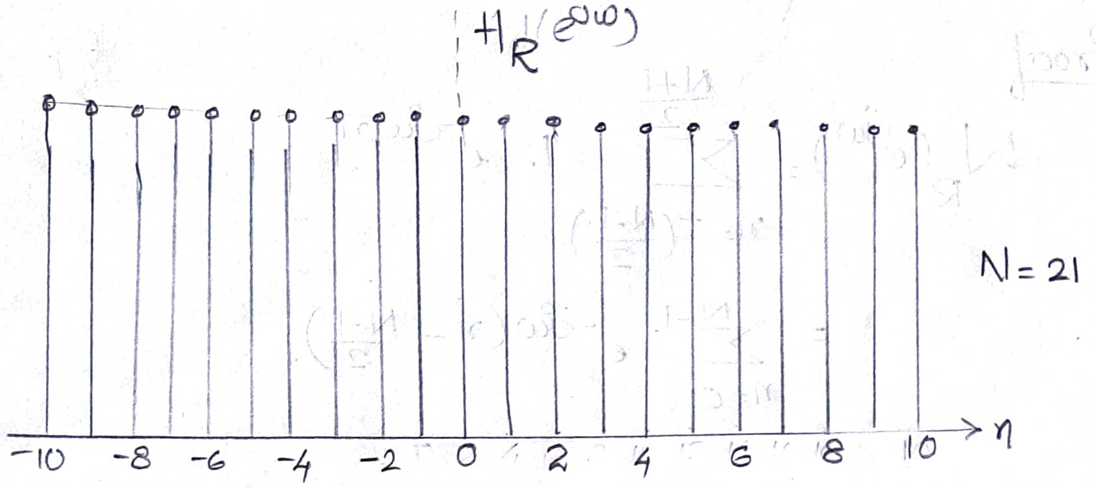
$$= e^{j\omega\left(\frac{N-1}{2}\right)} \left[\frac{e^{-j\omega\frac{N}{2}} \left(e^{j\omega N/2} - e^{-j\omega N/2} \right)}{e^{-j\omega/2} \left(e^{j\omega/2} - e^{-j\omega/2} \right)} \right]$$

$$W_R(e^{j\omega}) = \frac{2j \sin \omega\frac{N}{2}}{2j \sin \frac{\omega}{2}} = \frac{\sin \omega\frac{N}{2}}{\sin \frac{\omega}{2}}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

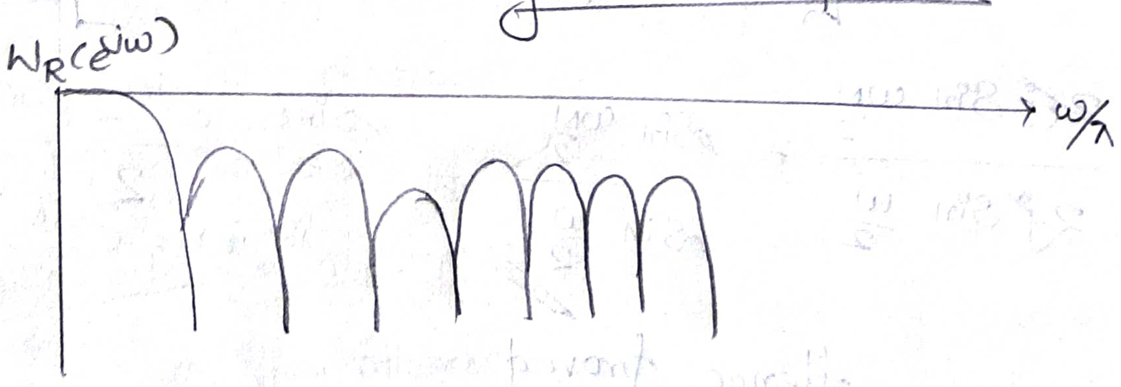
$$j2\sin\theta = e^{j\theta} - e^{-j\theta}.$$

Hence proved



Rectangular Window

Magnitude Response



FIR Filter design using windows.

Pm.

Q1. Design a linear phase FIR low pass filter using rectangular window by taking 7 samples of window sequence and with a cut-off frequency $\omega_c = 0.2\pi$ rad/sample

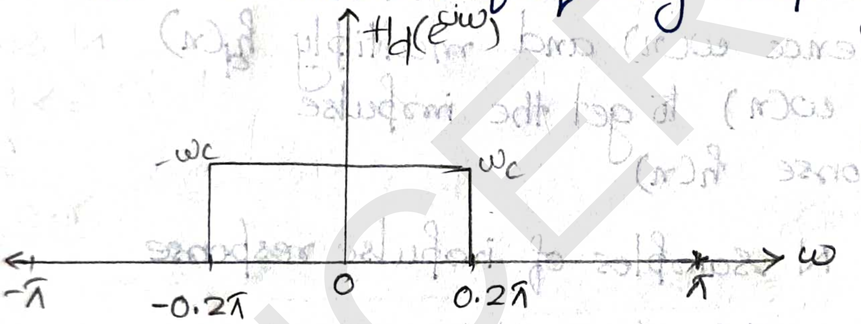
(OR)

Design a linear phase FIR low pass filter with freq. response

$$H_d(e^{j\omega}) = \begin{cases} 1 & -\omega_c \le \omega \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

where, $\omega_c = 0.2\pi$ rad/sample and $N = 7$ $\quad H_d(e^{j\omega}) = Ce^{-j\alpha\omega}$, $\alpha = 0$

Soln. Step 1: Plot the desired frequency response.



$\alpha = 0$. $h(n) = h(-n)$ (Case 5).

Step 2: Determine the desired impulse response $h_d(n)$ by taking F.T of $H_d(e^{j\omega})$.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) \cdot e^{j\omega n}\, d\omega \quad \text{(IFT Formula)}$$

$$e^{j\theta} - e^{-j\theta} = 2j \sin\theta$$

$$\Rightarrow h_d(n) = \frac{1}{2\pi}\int_{-0.2\pi}^{0.2\pi} 1 \cdot e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-0.2\pi}^{+0.2\pi} = \frac{1}{2\pi jn}\left[e^{j(0.2\pi)n} - e^{-j0.2\pi n}\right]$$

$$= \frac{1}{2\pi jn} 2j \sin(0.2\pi n)$$

$$h_d(n) = \frac{\sin(0.2\pi n)}{\pi n} \quad \text{for all } n \text{ except, } n=0$$

When $n=0$, apply limit.

L' Hospital Rule

$$\lim_{n \to 0} \frac{\sin 0.2\pi n}{\pi n}$$

$$\lim_{n \to 0} \frac{\sin 0.2\pi n}{0.2\pi n} \times 0.2 \quad (\text{OR})$$

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

$$= 0.2 \left(\lim_{n \to 0} \frac{\sin 0.2\pi n}{0.2\pi n} \right)$$

1

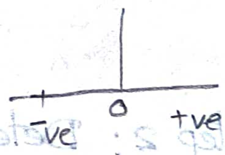
$h_d(n) = 0.2 \times 1 = \underline{0.2}$ for $n=0$

Step 3 : Choose a desired window sequence $\omega(n)$ and multiply $h_d(n)$ with $\omega(n)$ to get the impulse response $h(n)$

infinite to finite N samples
⇒ Multiply with a window

Calculate N samples of impulse response for $N = -\left(\frac{N-1}{2}\right)$ to $+\left(\frac{N-1}{2}\right)$

$n=0$ $\alpha=0$.



non cosal System ⇒ -ve values

$$W_R = \begin{cases} 1 & \text{for } n = -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

Here $N=7$ $\quad \frac{N-1}{2} = \frac{7-1}{2} = 3$

$$\therefore W_R = \begin{cases} 1 & \text{for } -3 \text{ to } 3 \\ 0 & \text{otherwise} \end{cases}$$

∴ Impulse response $h(n) = h_d(n) \times W_R(n)$.

$$h(n) = h_d(n) \text{ for } n = -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right)$$

Here $h(n) = h_d(n)$ for $n = -3$ to 3

From frequency response it is found that $\alpha = 0$

$\therefore$ We get a non-causal filter coefficients symmetrical about $n=0$.
(-ve values also)

$$h_d(n) = h_d(-n)$$

$$\therefore \boxed{h(n) = h(-n)} \quad \alpha = 0$$

Since $h(n)$ satisfies the symmetric condition

$$h(n) = h(-n).$$

Calculate $h(n)$ for $n = 0$ to 3

When $n=0$, $h(0) = h_d(0) \cdot W_R(0)$. → (1 value from qn.)

$$= 0.2 \times 1 = 0.2$$

$$h_d(n) = \frac{\sin 0.2\pi n}{\pi n}$$

$n=1$, $h(1) = h_d(1) \cdot W_R(1)$ (Calculate in radian mode)

$$= \frac{\sin 0.2\pi}{\pi} = \frac{\sin(0.628)}{\pi}$$

$$= \frac{0.587}{\pi} = \underline{0.1871}$$

$n=2$, $h(2) = h_d(2) \cdot W_R(2)$

$$= \frac{\sin(0.2\pi \times 2)}{2\pi} = \underline{0.1514}$$

$n=3$, $h(3) = h_d(3) \cdot W_R(3)$

$$= \frac{\sin(0.2\pi \times 3)}{3\pi} = \underline{0.1009}$$

$n=-1$, $h(-1) = h(1) = 0.1871$

$n=-2$, $h(-2) = h(2) = 0.1514$

$n=-3$, $h(-3) = h(3) = 0.1009$

Non-causal filter → Causal filter.

Step 4: Take the z-transform of $h(n)$ to get transfer fn.

$H(z)$ of FIR Low pass filter gn by.

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n) z^{-n}$$

Here $\left(\frac{N-1}{2}\right) = 3.$

$$\therefore H(z) = z^{-3} \sum_{n=-3}^{3} h(n) z^{-n}$$

$$= z^{-3}\left[h(-3) z^{-(-3)} + h(-2) z^{-(-2)} + h(-1) z^{-(-1)} + h(0) z^{0} + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3}\right]$$

Non-Causal
Not possible
z^{+} value - future

$$= z^{-3}\left[h(-3) z^{3} + h(-2) z^{2} + h(-1) z^{1} + h(0) z^{0} + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3}\right].$$

Causal ⇒

$$= h(-3) z^{0} + h(-2) z^{-1} + h(-1) z^{-2} + h(0) z^{-3} + h(1) z^{-4} + h(2) z^{-5} + h(3) z^{-6}].$$

$$= 0.1009 z^{0} + 0.1514 z^{-1} + 0.1871 z^{-2} + 0.2 z^{-3} + 0.1871 z^{-4} + 0.1514 z^{-5} + 0.1009 z^{-6}.$$

$$= 0.1009(z^{0} + z^{-6}) + 0.1514(z^{-1} + z^{-5}) + 0.1871(z^{-2} + z^{-4}) + 0.2 z^{-3}$$

$h(0) = 0.2$

~~$h(-1) = 0.1871$~~ $h(1) = 0.1871$ $h(4) = 0.1871$

~~$h(-2) = 0.1514$~~ $h(2) = 0.1514$ $h(5) =$

~~$h(-3) = 0.1009$~~ $h(3) = 0.1009$

May ask

Step 5 :- Find the frequency response of the filter when the impulse response is symmetric and N=odd with centre of symmetry at n=0 ($\alpha = 0$).

Magnitude response gn. by.

$$H(e^{j\omega}) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n.$$

$$= h(0) + \sum_{n=1}^{3} 2h(n)\cos\omega n$$

$$H(e^{j\omega}) = h(0) + 2h(1)\cos\omega + 2h(2)\cos 2\omega + 2h(3)\cos 3\omega$$

$$= 0.2 + 2\times 0.1871\cos\omega + 2\times 0.1514\cos 2\omega + 2\times 0.1009\cos 3\omega$$

$$= 0.2 + 0.3742\cos\omega + 0.3028\cos 2\omega + 0.2018\cos 3\omega$$

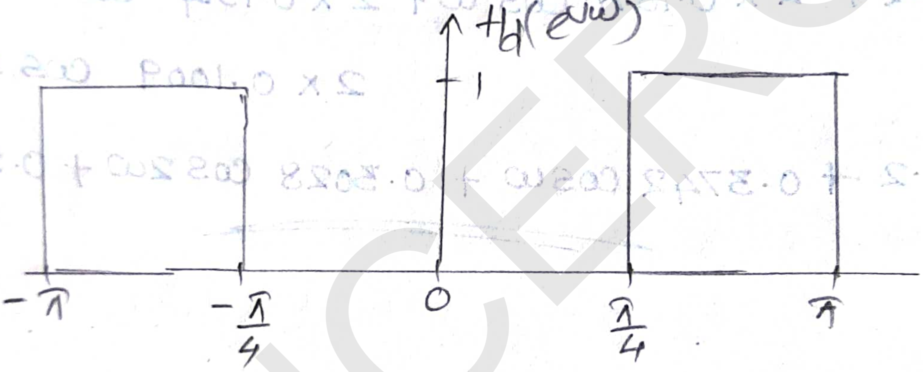
2. Design a Linear phase FIR High pass filter with frequency response

$$H_d(e^{j\omega}) = 1 \quad \text{for } \frac{\pi}{4} \le |\omega| \le \pi$$
$$\qquad 0 \quad \text{for } |\omega| < \frac{\pi}{4}$$

Find the value of $h(n)$ for $N=11$. Find $H(z)$. Use rectangular window.

Soln. Step 1: Plot the desired frequency response.

$$H_d(e^{j\omega}) = 1 \quad \text{for } \frac{\pi}{4} \le |\omega| \le \pi$$
$$\qquad 0 \quad \text{for } |\omega| < \frac{\pi}{4}$$



Here $H_d(e^{j\omega}) = 1 \quad \therefore \alpha = 0$ $\qquad$ $(c e^{-j\alpha\omega})$

$c = 1 \;\therefore\; \alpha = 0$

When $\alpha = 0$

impulse response is symmetric

at $n=0$ $h(n) = h(-n)$

Step 2: Find inverse fourier transform

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{-\pi/4} 1 \cdot e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{+\pi/4}^{\pi} 1 \cdot e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{-\pi}^{-\pi/4} + \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{\pi/4}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\frac{\pi}{4}n} - e^{-j\pi n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\pi n} - e^{+j\pi/4 n}}{jn}\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\frac{\pi}{4}n} - e^{+j\pi/4 n} + (e^{+j\pi n} - e^{-j\pi n})}{jn}\right] = \frac{-1}{2\pi}\left[(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}) - (e^{j\pi n} - e^{-j\pi n})\right]$$

$$e^{j\theta} - e^{-j\theta} = 2j\sin\theta$$

$$= -\frac{1}{2\pi jn}\left[2j\sin\frac{\pi}{4}n - 2j\sin\pi n\right]$$

$$h_d(n) = -\frac{1}{2\pi jn}\left(2j\sin\frac{\pi}{4}n - 2j\sin\pi n\right)$$

$$= -\frac{2j}{2\pi jn}\left(\sin\frac{\pi}{4}n - \sin\pi n\right)$$

$$h_d(n) = \frac{\sin\pi n - \sin\frac{\pi}{4}n}{\pi n}$$

$$h_d(n) = \frac{\sin\pi n - \sin\frac{\pi}{4}n}{\pi n} \quad \text{for all } n \text{ except } n=0$$

L' Hospital Rule.

For $n=0$

$$\lim_{n\to 0}\frac{\sin\pi n}{\pi n} - \lim_{n\to 0}\frac{\sin\pi/4\, n}{\pi n}$$

$$\lim_{\theta\to 0}\frac{\sin A\theta}{\theta}$$

$$= \frac{1}{\pi}\lim_{n\to 0}\frac{\sin\pi n}{n} - \frac{1}{\pi}\lim_{n\to 0}\frac{\sin\pi/4\, n}{n}$$

$$= \frac{1}{\pi}\cdot\pi - \frac{1}{\pi}\cdot\frac{\pi}{4}$$

$$= 1 - \frac{1}{4} = \frac{3}{4} \qquad h_d(0) = \frac{3}{4}$$

Rectangular Window

$$\omega_R(n) = \begin{cases} 1 & \text{for } \left(-\frac{N-1}{2} \le n \le \frac{N-1}{2}\right) \\ 0 & \text{Otherwise.} \end{cases}$$

Step 3: Multiply with $h_d(n)$ to make it a finite impulse response.

$\therefore h(n) = h_d(n) \times W_R(n)$. for $-\left(\frac{N-1}{2}\right) \le n \le \left(\frac{n-1}{2}\right)$

Gn. N=1

Here α $\frac{N-1}{2} = \frac{11-1}{2} = 5$

$h_d(n) = h_d(n) \times W_R(n)$ for $-5 \le n \le 5$

$h(n) = h(-n)$ (Symmetric Condition)

$h(0) = h_d(0) \cdot W_R(0)$.

$h(0) = \frac{3}{4} \cdot 1 \Rightarrow \boxed{h(0) = \frac{3}{4}}$

$h(1) = h_d(1) \cdot W_R(0)$

$= \frac{\sin \pi - \sin \frac{\pi}{4}}{\pi} \times 1 \Rightarrow \frac{1}{\pi}(0 - 0.707)$

$= \frac{-0.707}{3.14} = -0.2250$

$\therefore \boxed{h(1) = h(-1) = -0.2250}$

$h(2) = \frac{\sin 2\pi - \sin \frac{2\pi}{4}}{2\pi} \times 1 = \frac{0-1}{2\pi}$

$= -0.1591$ $h(2) = h(-2) = -0.1591$

$$h(3) = \frac{\sin 3\pi - \sin\frac{3\pi}{4}}{3\pi} \times 1 = \frac{0 - 0.7071}{3\pi}$$

$$= -0.0750$$

$$\boxed{h(3) = h(-3) = -0.0750}$$

$$h(4) = \frac{\sin 4\pi - \sin\frac{4\pi}{4}}{4\pi} \times 1 = \frac{0-0}{4\pi} = 0$$

$$\boxed{h(4) = h(-n) = 0}$$

$$h(5) = \frac{\sin 5\pi - \sin\frac{5\pi}{4}}{5\pi} \times 1 = \frac{0 - (-0.7071)}{5\pi}$$

$\sin\frac{5\pi}{4} = -0.7071$

$$= 0.0450$$

$$\boxed{h(5) = h(-5) = 0.0450}$$

Step 4: Transfer function of filter given by.

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n) z^{-n}$$

Here $\frac{N-1}{2} = 5$.

$$H(z) = z^{-5} \sum_{n=5}^{5} h(n) z^{-n}$$

$$= z^{-5}\Big[h(-5)z^{5} + h(-4)z^{4} + h(-3)z^{3} + h(-2)z^{2} + h(-1)z^{1} + h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5}\Big].$$

$$H(z) = \Big[h(-5)z^{0} + h(-4)z^{-1} + h(-3)z^{-2} + h(-2)z^{-3} + h(-1)z^{-4} + h(0)z^{-5} + h(1)z^{-6} + h(2)z^{-7} + h(3)z^{-8} + h(4)z^{-9} + h(5)z^{-10}\Big]$$

$$H(z) = 0.0450z^{0} + 0z^{-1} + (-0.0750z^{-2}) + (-0.1591z^{-3}) + (-0.2250z^{-4}) + 0.75z^{-5} + (-0.2250z^{-6}) + (-0.1591z^{-7}) + -0.0750z^{-8} + 0 \times z^{-9} + 0.0450 \times z^{-10}$$

$h(0) = 0.0450$

$h(1) = 0$

$h(2) = -0.0750$

$h(3) = -0.1591$

$h(4) = -0.2250$

$h(5) = 0.75$

$h(6) = -0.2250$

$h(7) = ~~-0.075~~ -0.1591$

$h(8) = -0.0750$

$h(9) = 0$

$h(10) = 0.0450$

$H(z) = 0.045 z^0 + 0 z^{-1} + (-0.075 z^{-2}) + (-0.159 z^{-3}) + (-0.225 z^{-4}) + 0.75 z^{-5} + (-0.225 z^{-6}) + (-0.159 z^{-7}) + -0.075 z^{-8} + 0 \times z^{-9} + 0.045 \times z^{-10}$

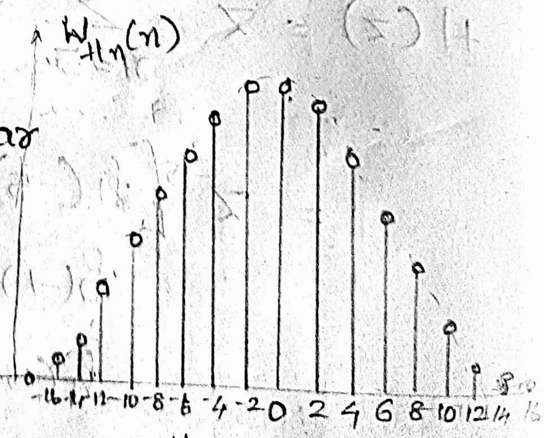
$h(0) = 0.045$
$h(1) = 0$
$h(2) = -0.075$
$h(3) = -0.159$
$h(4) = -0.225$
$h(5) = 0.75$
$h(6) = -0.225$
$h(7) = ~~-0.075~~ -0.159$
$h(8) = -0.075$
$h(9) = 0$
$h(10) = 0.045$

HANNING WINDOW

Hanning window sequence

$$w_{Hn}(n) = \begin{cases} 0.5 + 0.5 \cos \frac{2\pi n}{N-1} & \text{for } -\left(\frac{N-1}{2}\right) \text{ to } +\left(\frac{N-1}{2}\right) \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

↳ The main lobe width
  ⇒ twice than Rectangular window
  → Results in doubling of the transition region of the filter.



Hanning Window Sequence.

↳ magnitude of side lobe ⇒ -31 dB.
  → 18 dB lower than Rect. Window.

↳ Smaller ripples in both pass band and stop band of the LPF designed using Hanning Window.

Hanning window sequence

when $h(n) = h(-n)$ $\quad \alpha = 0$
Symmetric Cond^n

Then

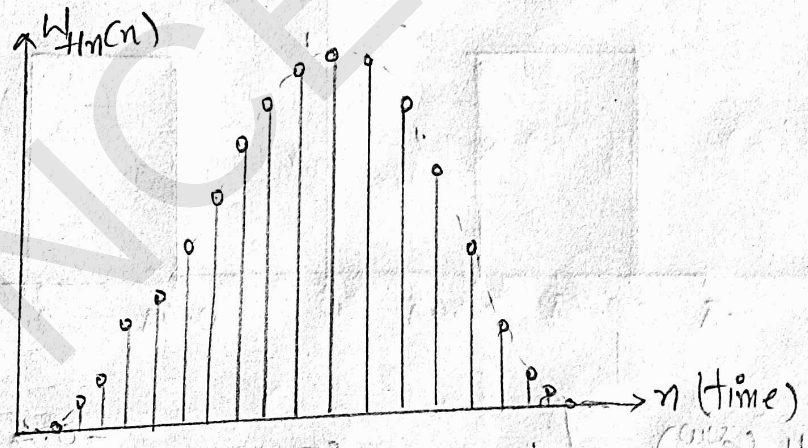
(i) $$w_{Hn}(n) = \begin{cases} 0.5 + 0.5\cos\dfrac{2\pi n}{N-1} & \text{for } -\left(\dfrac{N-1}{2}\right) \text{ to } \left(\dfrac{N-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

↑ N ⇒ +ve/−ve Values.

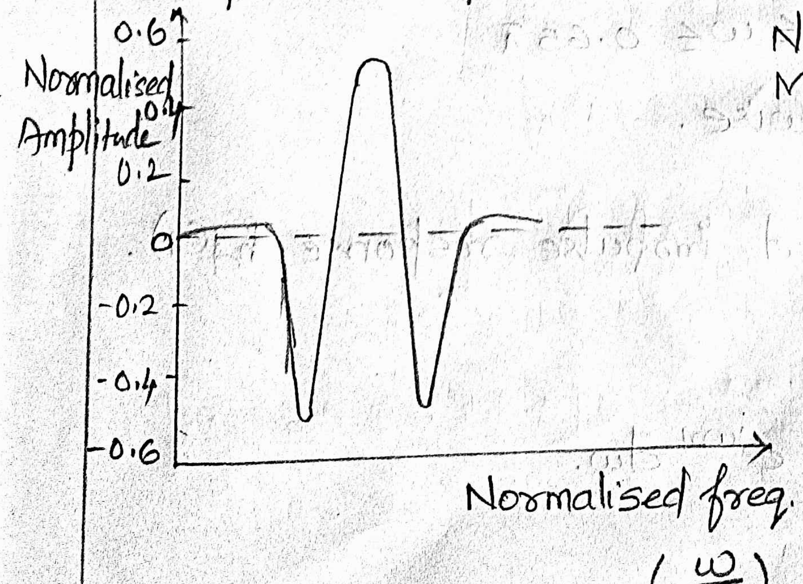
(ii) when $h(n) = h(N-1-n)$ Symmetric Cond^n

$$w_{Hn}(n) = \begin{cases} 0.5 - 0.5\cos\dfrac{2\pi n}{N-1} & \text{for } n = 0 \text{ to } N-1 \\ 0 & \text{otherwise} \end{cases}$$

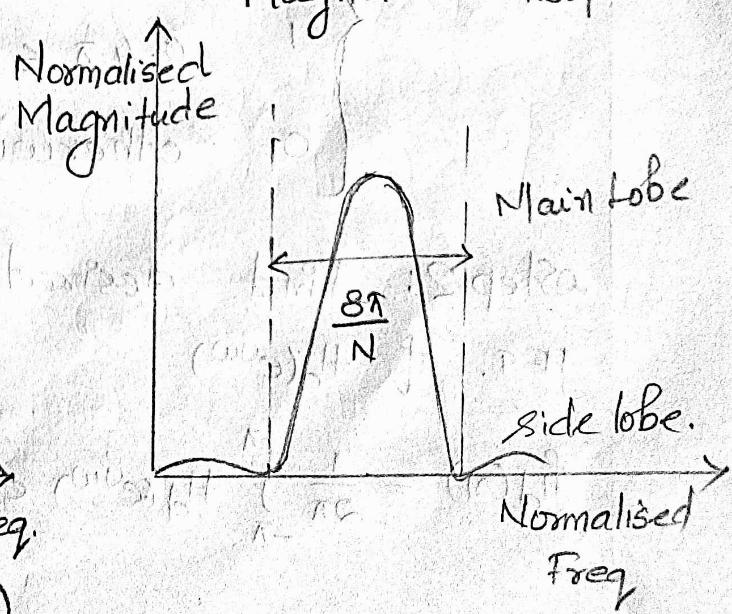
← N ⇒ +ve Values



Amplitude response



Magnitude Response.



6m.

Qn. Design a linear phase FIR bandpass filter to pass frequencies in the range 0.4π to 0.65π rad/sample by taking 7 samples of hanning window sequence.

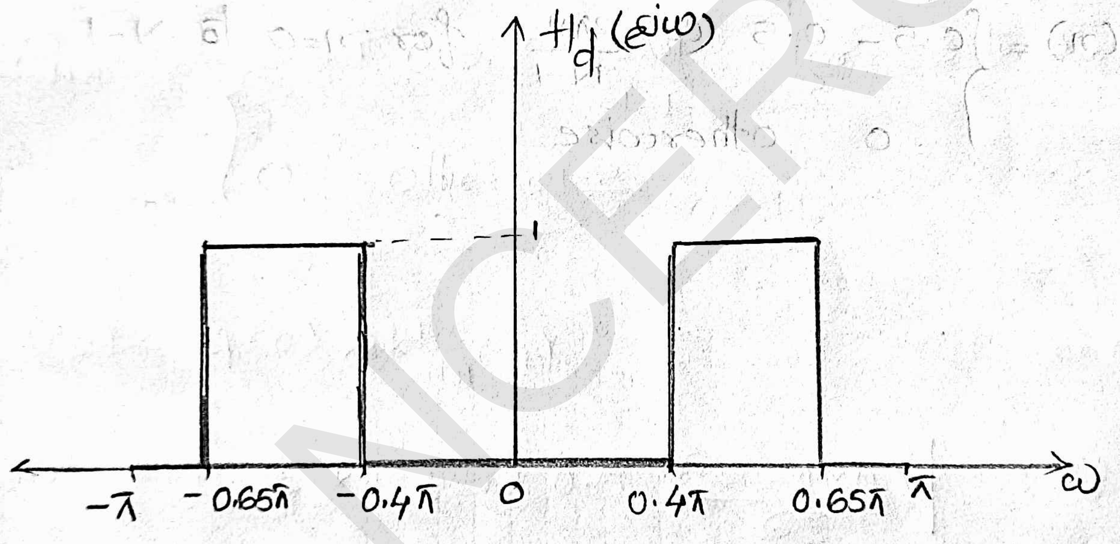
Soln. Gn ⇒ FIR band pass filter

Freq. range ⇒ 0.4π to 0.65π

$N = 7$

Window ⇒ Hanning Window.

Step 1: Plot desired frequency response.



$$H_d(e^{j\omega}) = \begin{cases} 1 & -0.65\pi \le \omega \le -0.4\pi \\ 1 & 0.4\pi \le \omega \le 0.65\pi \\ 0 & \text{otherwise.} \end{cases}$$

Step 2: Find desired impulse response. $h_d(n)$.

IFT. of $H_d(e^{j\omega})$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega.$$

$$= \frac{1}{2\pi}\int_{-0.65\pi}^{-0.4\pi} e^{j\omega n}\,d\omega + \frac{1}{2\pi}\int_{0.4\pi}^{0.65\pi} e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-0.65\pi}^{-0.4\pi} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{0.4\pi}^{0.65\pi}$$

$$= \frac{1}{2\pi}\left(\frac{e^{-j0.4\pi n} - e^{-j0.65\pi n}}{jn}\right) + \frac{1}{2\pi}\left(\frac{e^{j0.65\pi n} - e^{j0.4\pi n}}{jn}\right)$$

$$= \frac{1}{2\pi jn}\left[\frac{e^{-j0.4\pi n} - e^{+j0.4\pi n} - e^{-j0.65\pi n} + e^{j0.65\pi n}}{}\right)$$

$$= -\frac{1}{2\pi jn}\left(e^{j0.4\pi n} - e^{-j0.4\pi n}\right) - \left(e^{j0.65\pi n} - e^{-j0.65\pi n}\right)$$

$$= -\frac{1}{2\pi jn}\left(2j\sin 0.4\pi n - 2j\sin 0.65\pi n\right)$$

$$= -\frac{2j}{2\pi jn}\left(\sin 0.4\pi n - \sin 0.65\pi n\right)$$

$$\Rightarrow h_d(n) = \left(\frac{\sin 0.65\pi n - \sin 0.4\pi n}{\pi n}\right) \quad \text{for all } n \text{ except } n=0.$$

$$\lim_{\theta\to 0}\frac{\sin A\theta}{\theta} = A$$

$h_d(n)$ for $n=0$.

$$h_d(0) = \lim_{n\to 0}\frac{\sin 0.65\pi n}{\pi n} - \lim_{n\to 0}\frac{\sin 0.4\pi n}{\pi n}$$

$$= \frac{1}{\pi}\lim_{n\to 0}\frac{\sin 0.65\pi n}{n} - \frac{1}{\pi}\lim_{n\to 0}\frac{\sin 0.4\pi n}{n}$$

$$= \frac{1}{\pi}\times 0.65\pi - \frac{1}{\pi}\times 0.4\pi = 0.65 - 0.4$$

$$h_d(0) = 0.25$$

Step 3: Multiply $h_d(n)$ with $w_{Hn}(n)$ to get $h(n)$

Since $\alpha = 0$ the symmetric condition is $h(n) = h(-n)$

$h(n) = h_d(n) \times W_{Hn}(n)$ for $-\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$.

Here $N = 7$ $\frac{N-1}{2} = 3$.

$h(n) = h_d(n) \times w_{Hn}(n)$ for ~~-3 to 3~~ $-3 \le n \le 3$

$h(0) = h_d(0) \times w_{Hn}(0)$

$$W_{Hn}(n) = \begin{cases} 0.5 + 0.5 \cos\frac{2\pi n}{N-1} & \text{for } -3 \text{ to } 3. \\ 0 & \text{otherwise} \end{cases}$$

$h(0) = h_d(0) * W_{Hn}(0)$ $\cos 0 = 1$

$= 0.25 * \left[0.5 + 0.5 \cos\left(\frac{2\pi \times 0}{6}\right)\right]$

$= 0.25 \times [0.5 + 0.5) = 0.25 \times 1$

$h(0) = 0.25$

$h(1) = h_d(1) * W_{Hn}(1)$

$\left(\frac{\sin(0.65\pi n - \sin 0.4\pi n}{\pi n}\right) \times \left[0.5 + 0.5\cos\left(\frac{2\pi n}{N-1}\right)\right]$

$= \left(\frac{\sin 0.65\pi - \sin 0.4\pi}{\pi}\right) \times \left[0.5 + 0.5 \cos\left(\frac{2\pi}{6_3}\right)\right]$

$$= \left(\frac{\sin(2.042) - \sin(1.25)}{\pi} \right) \times \left[0.5 + 0.5 \cos\left(\frac{\pi}{3}\right) \right]$$

$$= \left(\frac{0.891 - 0.95}{\pi} \right) \times \left(0.5 + 0.5 \cos(1.047) \right)$$

$$= \left(\frac{-0.06}{\pi} \right) \times 0.5 + 0.5 \times 0.5$$

$$= -0.019 \times (0.5 + 0.25)$$

$$= -0.019 \times 0.75$$

$$h(1) = -0.0143$$

$n=2$, $$h(2) = \left(\frac{\sin 0.65\pi \times 2 - \sin 0.4\pi \times 2}{\pi \times 2} \right) \left[0.5 + 0.5 \cos\left(\frac{\pi \times 2}{3}\right) \right]$$

$$h(2) = -0.0556$$

$$h(3) = \left(\frac{\sin 0.65\pi \times 3 - \sin 0.4\pi \times 3}{\pi \times 3} \right) \left(0.5 + 0.5 \cos\left(\frac{\pi \times 3}{3}\right) \right)$$

$$h(3) = 0$$

Using symmetric condition.

$$h(-1) = h(1) = -0.0143$$

$$h(-2) = h(2) = -0.0556$$

$$h(-3) = h(3) = 0$$

$$h(0) = 0.25$$

Step 4: Find transfer function $H(z)$

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n) z^{-n}$$

$$= z^{-3}\left[h(-3)z^{3} + h(-2)z^{2} + h(-1)z^{1} + h(0)z^{0} + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}\right]$$

$$= h(-3)z^{0} + h(-2)z^{-1} + h(-1)z^{-2} + h(0)z^{-3} + h(1)z^{-4} + h(2)z^{-5} + h(3)z^{-6}.$$

$$H(z) = 0 \times z^{0} + (-0.556z^{-1}) + (-0.0143z^{-2}) + 0.25z^{-3} + -0.0143z^{-4} + -0.0556z^{-5} + 0 \times z^{-6}.$$

$h(0) = 0$ $\quad$ $h(4) = -0.0143$

$h(1) = -0.556$ $\quad$ $h(5) = -0.0556$

$h(2) = -0.0143$ $\quad$ $h(6) = 0$

$h(3) = 0.25$

~~HAMMING WINDOW~~

HAMMING WINDOW

Equation for Hamming Window,

$$W_H(n) = \begin{cases} 0.54 + 0.46 \cos \frac{2\pi n}{N-1} & \text{for } n = -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right) \\ 0 & \text{for other } n \end{cases}$$

$(\alpha = 0)$

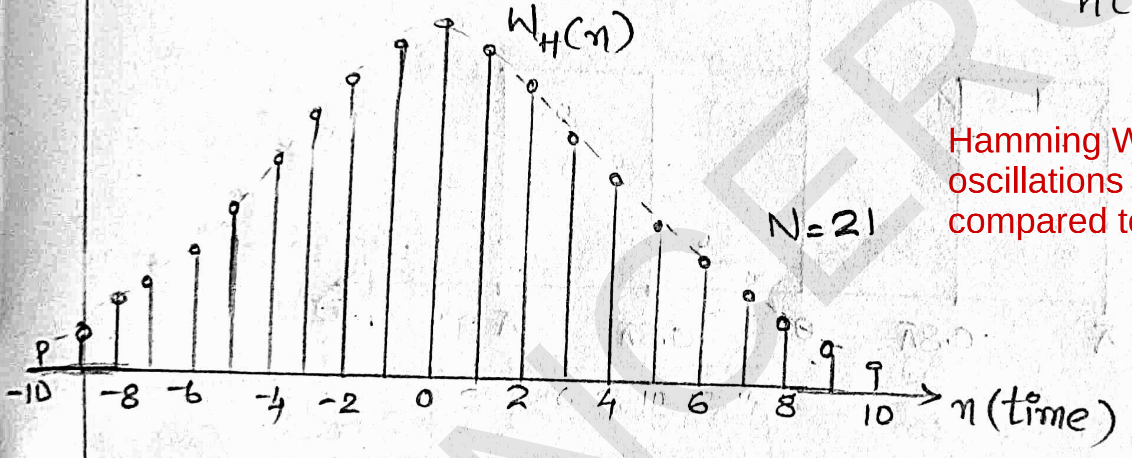
$h(n) = h(-n)$

(OR)

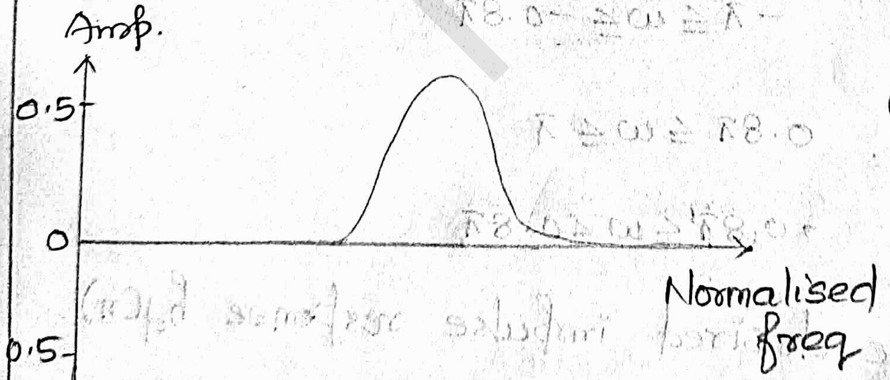
$$W_H(n) = \begin{cases} 0.54 - 0.46 \cos \frac{2\pi n}{N-1} & ; \text{ for } n = 0 \text{ to } N-1 \\ 0 & , \text{ other } n. \end{cases}$$

$\Rightarrow \alpha = \text{Value}.$

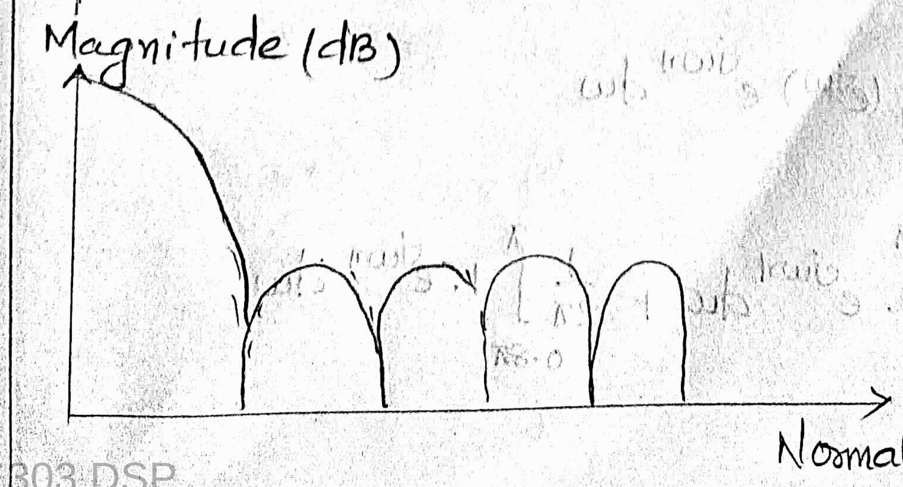
$h(n) = h(N-1-n).$



Hamming Window side lobe oscillations are lesser when compared to Hanning Window



(Freq Response)



(Magnitude Response)

Qn. Design a linear phase FIR high pass filter using hamming Window, with a cut-off frequency $\omega_c = 0.8\pi$ rad/sample and $N=7$

Soln. Given

- FIR Highpass filter
- Cut-off frequency $= 0.8\pi$ rad/sample
- $N=7$

Step 1 : Plot the desired frequency range.

$$H_d(e^{j\omega}) = \begin{cases} 1 & -\pi \le \omega \le -0.8\pi \\ 1 & 0.8\pi \le \omega \le \pi \\ 0 & \text{otherwise} \end{cases}$$

Step 2:- Find the desired impulse response $h_d(n)$.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{-0.8\pi} 1 \cdot e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{0.8\pi}^{\pi} 1 \cdot e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-0.8\pi} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{0.8\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j0.8\pi n} - e^{-j\pi n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\pi n} - e^{j0.8\pi n}}{jn}\right].$$

$$= \frac{1}{2\pi jn}\left[e^{-j0.8\pi n} - e^{j0.8\pi n} - e^{-j\pi n} + e^{j\pi n}\right]$$

$$= -\frac{1}{2\pi jn}\left[\left(e^{j0.8\pi n} - e^{-j0.8\pi n}\right) - \left(e^{j\pi n} - e^{-j\pi n}\right)\right]$$

$$= -\frac{1}{2\pi jn}\left(2j \sin 0.8\pi n - 2j \sin \pi n\right)$$

$$= -\frac{2j}{2\pi jn}\left(\sin 0.8\pi n - \sin \pi n\right).$$

$$\boxed{h_d(n) = \frac{\sin \pi n - \sin 0.8\pi n}{\pi n}}$$ For all n, except $n=0$.

For $n=0$.

$$h_d(0) = \lim_{n\to 0} \frac{\sin \pi n - \sin 0.8\pi n}{\pi n}$$

$$= \frac{1}{\pi}\lim_{n\to 0}\frac{\sin \pi n}{n} - \frac{1}{\pi}\lim_{n\to 0}\frac{\sin 0.8\pi n}{n}$$

$$= \frac{1}{\pi}\times\pi - \frac{1}{\pi}\,0.8\pi = 1-0.8 = 0.2$$

$$\boxed{h_d(0) = 0.2}$$

Step 3 :- To find h(n)

$h(n) = h_d(n) \times W_H(n)$ $N=7.$

Symmetric condition ⇒ $h(n) = h(-n)$.

Hamming window,

$w_H(n) = 0.54 + 0.46 \cos\left(\frac{2\pi n}{N-1}\right)$ $n = -\frac{(N-1)}{2}$ to $\left(\frac{N-1}{2}\right)$

$N=7$ $\frac{N-1}{2} = \frac{7-1}{2} = 3.$

$\therefore W_H(n) = 0.54 + 0.46 \cos\left(\frac{2\pi n}{N-1}\right)$ $n = -3$ to $3.$

$n=0$ $h(0) = h_d(0) \times W_H(0)$

$= 0.2 \times \left[0.54 + 0.46 \cos\left(\frac{2\pi \times 0}{6}\right)\right]$

$= 0.2 \times [0.54 + (0.46 \times 1)]$

$= 0.2 \times 1$

$\boxed{h(0) = 0.2}$

$n=1$, $h(1) = h_d(1) \times W_H(1)$

$h_d(n) = \frac{\sin \pi n - \sin 0.8\pi n}{\pi n} = \boxed{\frac{\sin \pi - \sin 0.8\pi}{\pi}}$ ← $h_d(1)$

$h(1) = \left(\frac{\sin \pi - \sin 0.8\pi}{\pi}\right) \times \left(0.54 + 0.46 \cos\left(\frac{2\pi}{6}\right)\right)$

$= \left(\frac{0 - 0.5878}{\pi}\right) \times 0.54 + 0.46 \times 0.5$

$= -0.1871 \times 0.54 + 0.23 = -0.1871 \times 0.77$

$\boxed{h(1) = -0.1441} = h(-1)$

$n = 2, \quad h(2) = h_d(2) \times W_H(2)$

$$= \left(\frac{\sin 2\pi - \sin 0.8\pi \times 2}{2\pi}\right) \times \left(0.54 + 0.46 \cos\frac{4\pi}{6}\right)$$

$$= 0 - (-0.9511) \times 0.54 + (0.46 \times -0.5)$$

$$\frac{0.9511}{2\pi} \times 0.31 = 0.1513 \times 0.31 = 0.0469$$

$\boxed{h(2) = 0.0469 = h(-2)}$

$n = 3$

$h(3) = h_d(3) \times W_H(3)$

$$= \left(\frac{\sin 3\pi - \sin 0.8\pi \times 3}{3\pi}\right) \times \left[0.54 + 0.46 \cos\left(\frac{2\pi \times 3}{6}\right)\right]$$

$$= \frac{0 - (0.9511)}{3\pi} \times 0.54 + 0.46 \times -1$$

$$= -0.1009 \times 0.08 = -8.0731 \times 10^{-3} = -0.0081$$

$\boxed{h(3) = h(-3) = -0.0081}$

Step 4 :- To find transfer function H(z)

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} h(n) z^{-n}$$

$N = 7$

$\frac{N-1}{2} = 3.$

$$H(z) = z^{-3}\left[h(-3) z^{3} + h(-2) z^{2} + h(-1) z^{1} + h(0) z^{0} + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3}\right]$$

$$H(z) = \left[h(-3) z^{0} + h(-2) z^{-1} + h(-1) z^{-2} + h(0) z^{-3} + h(1) z^{-4} + h(2) z^{-5} + h(3) z^{-6}\right]$$

$$H(z) = -0.0081 z^{0} + 0.0469 z^{-1} + (-0.1441) z^{-2} + 0.2 z^{-3} + (-0.1441) z^{-4} + 0.0469 z^{-5} + (-0.0081) z^{-6}$$

$h(0) = -0.0081$ $\quad$ $h(4) = -0.1441$

$h(1) = 0.0469$ $\quad$ $h(5) = 0.0469$

$h(2) = -0.1441$ $\quad$ $h(6) = -0.0081$

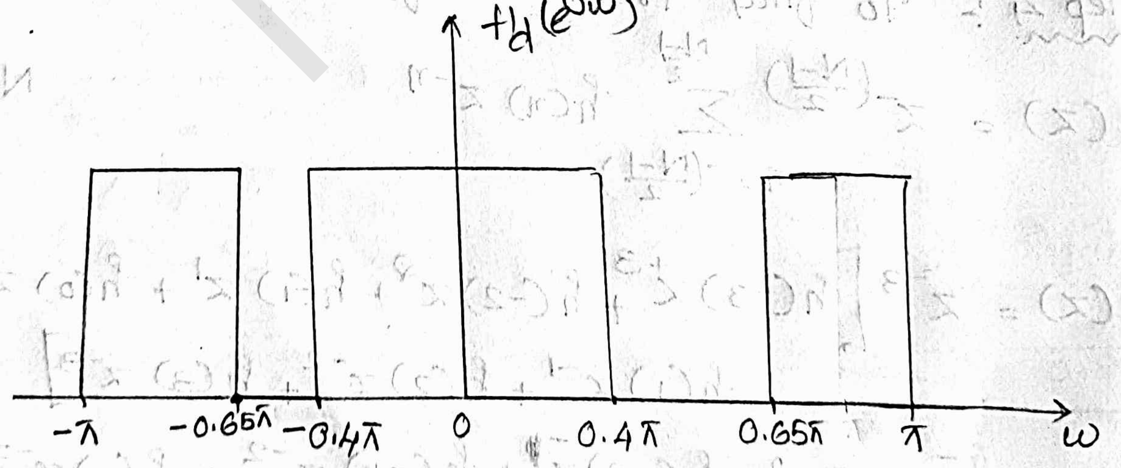
$h(3) = 0.2$

Qn. Design a Linear phase FIR bandstop filter to reject frequencies in the range 0.4π to 0.6π rad/Sample using rectangular window by taking 7 samples of window sequence.

Soln. Given data

- To design band stop filter.
- frequency, $\Rightarrow 0.4\pi$ to 0.6π rad/Sample.
- $N = 7$
- Rectangular Window.

Step 1: To ~~find~~ Plot the desired frequency respons.



$$H_d(e^{j\omega}) = \begin{cases} 1, & -\pi \le \omega \le -0.65\pi \\ 1, & -0.4\pi \le \omega \le 0.4\pi \\ 1, & 0.65\pi \le \omega \le \pi \\ 0, & \text{otherwise} \end{cases}$$

~~$H_d(e^{j\omega})$~~

Step 3: Find $h_d(n)$ desired impulse response.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-0.65\pi} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{-0.4\pi}^{0.4\pi} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{0.65\pi}^{\pi} e^{j\omega n} d\omega.$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-0.65\pi} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-0.4\pi}^{0.4\pi} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{0.65\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j0.65\pi n} - e^{-j\pi n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j0.4\pi n} - e^{-j0.4\pi n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\pi n} - e^{j0.65\pi n}}{jn}\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j0.65\pi n} - e^{-j\pi n} + e^{j0.4\pi n} - e^{-j0.4\pi n} + e^{j\pi n} - e^{j0.65\pi n}}{jn}\right]$$

~~$= \frac{1}{2\pi jn}\; e^{-j0.65\pi n} - e^{j0.65\pi n} + e$~~

$$= \frac{1}{2\pi jn}\left[(e^{j\pi n} - e^{-j\pi n}) + (e^{j0.4\pi n} - e^{-j0.4\pi n}) - (e^{j0.65\pi n} - e^{-j0.65\pi n})\right]$$

$$= \frac{1}{2\pi jn}\left[2j\sin \pi n + 2j \sin 0.4\pi n - 2j \sin 0.65\pi n\right]$$

$$= \frac{2j}{2\pi jn}\left[\sin \pi n + \sin 0.4\pi n - \sin 0.65\pi n\right]$$

$$h_d(n) = \frac{\sin \pi n + \sin 0.4\pi n - \sin 0.65\pi n}{\pi n}$$

for all n, except n=0.

for n=0

$$h_d(0) = \lim_{n\to 0} \left(\frac{\sin \pi n + \sin 0.4\pi n - \sin 0.65\pi n}{\pi n}\right).$$

$$= \frac{1}{\pi}\lim_{n\to 0}\frac{\sin \pi n}{n} + \frac{1}{\pi}\lim_{n\to 0}\frac{\sin 0.4\pi n}{n} - \frac{1}{\pi}\lim_{n\to 0}\frac{\sin 0.65\pi n}{n}$$

$$= \frac{1}{\pi}\times\pi + \frac{1}{\pi}\times 0.4\pi - \frac{1}{\pi}\,0.65\pi$$

$$= 1 + 0.4 - 0.65 = 0.75$$

$$\boxed{h_d(0) = 0.75}$$

Step 3:- To find $h(n)$

$$h(n) = h_d(n) \times W_R(n).$$

Symmetric Cond^n.
$h(n) = h(-n)$

$$W_R(n) = \begin{cases} 1 & \text{for } -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right) \\ 0 & \text{for other} \end{cases}$$

$n=0$

$$h(0) = h_d(0) \times W_R(0)$$

$$= 0.75 \times 1$$

$$\boxed{h(0) = 0.75}$$

$$h(1) = h_d(1) \times W_R(1)$$

$$= \left(\frac{\sin\pi + \sin 0.4\pi - \sin 0.65\pi}{\pi}\right) \times 1$$

$$h(1) = \frac{0 + 0.9511 - 0.8910}{\pi} = 0.0191$$

$$\boxed{h(1) = h(-1) = 0.0191}$$

$$h(2) = h_d(2) \times W_R(2)$$

$$= \left(\frac{\sin 2\pi + \sin(0.4\pi \times 2) - \sin(0.65\pi \times 2)}{2\pi}\right) \times 1$$

$$= \frac{0 + 0.5878 - (-0.8090)}{2\pi} = 0.2223$$

$$\boxed{h(2) = h(-2) = 0.2223}$$

$$h(3) = h_d(3) \times W_R(3)$$

$$= \left(\frac{\sin 3\pi + \sin(0.4\pi \times 3) - \sin(0.65\pi \times 3)}{3\pi}\right) \times 1$$

$$= \frac{0 + (-0.5878) - (-0.1564)}{3\pi} = -0.0458$$

$$\boxed{h(3) = h(-3) = -0.0458}$$

Step 4 :- To find transfer fn $H(z)$.

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n)\, z^{-n} \qquad \frac{N-1}{2} = 3.$$

$$H(z) = z^{-3}\left[h(-3)z^{3} + h(-2)z^{2} + h(-1)z^{1} + h(0)z^{0} + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}\right]$$

$$= h(-3)z^{0} + h(-2)z^{-1} + h(-1)z^{-2} + h(0)z^{-3} + h(1)z^{-4} + h(2)z^{-5} + h(3)z^{-3}.$$

$$H(z) = -0.0458\, z^{0} + 0.2223 z^{-1} + 0.0191\, z^{-2} + 0.75\, z^{-3} + 0.0191\, z^{-4} + 0.2223 z^{-5} + (-0.0458) z^{-6}$$

$h(0) = -0.0458$

$h(1) = 0.2223$

$h(2) = 0.0191$

$h(3) = 0.75$

$h(4) = 0.0191$

$h(5) = 0.2223$

$h(6) = -0.0458$.

DESIGN OF FIR FILTERS BY FREQUENCY SAMPLING METHOD.

- 2nd design technique.
- Sampling the frequency.
- Frequency sampling - using DFT eqn.
- Window ⇒ $H_d(e^{j\omega})$ ⇒ IFT → analysed in Time domain itself.
- Frequency sampling ⇒ In Freq. domain → freq. spectrum Sampled (DFT).

  Take Inverse From (DFT) → Transfer fn.

- In this method, the ideal (desired) frequency response is sampled at sufficient no. of points (N points)
- These samples are the Discrete Fourier Transform (DFT) Coefficients of impulse response of filter.
- Hence impulse response of filter is determined by taking Inverse DFT.
- Let $H_d(e^{j\omega})$ = Ideal desired frequency response

  $H(k)$ = DFT sequence obtained by sampling $H_d(e^{j\omega})$

  $h(n)$ = Impulse response of FIR Filter.

Procedure :-

step 1 ⇒ Choose the ideal (desired) frequency response $H_d(e^{j\omega})$

step 2 ⇒ Sample $H_d(e^{j\omega})$ at N points by taking $\omega = \omega_k = \frac{2\pi k}{N}$ where $k = 0, 1, 2, 3 \cdots (N-1)$ to generate the sequence $H(k)$.

$$\therefore H(k) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}}$$

for $k = 0, 1, 2, \ldots N-1$

step 3 ⇒ Compute the N samples of impulse response $h(n)$ using following eqn.

when N is odd.

Impulse response.

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\left(\frac{N-1}{2}\right)} Re\left[H(k)\, e^{\frac{j2\pi nk}{N}}\right]\right]$$

When N is even

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N}{2}-1} Re\left[H(k)\, e^{\frac{j2\pi nk}{N}}\right]\right]$$

Re stands for real part of.

step 4 :- Z - transform of impulse response $h(n)$ to get filter transfer fn. $H(z)$.

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

pbm

Qn. Design a linear phase FIR LPF with cut-off frequency of 0.5π rad/sample by taking 11 samples of ideal frequency response.

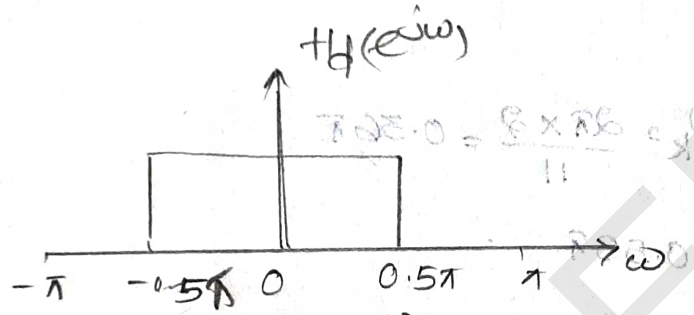
Soln Given – Method → Frequency Sampling.

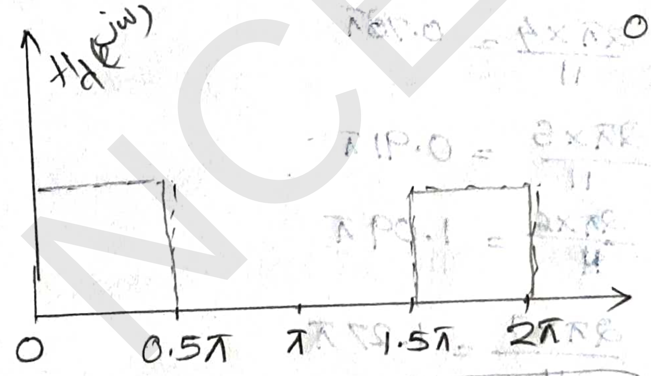
↳ FIR LPF

$\omega_c \Rightarrow 0.5\pi$ rad/sample.

N=1

Step 1





Step 2:- Sample $H_d(e^{j\omega})$ at N points

$$\omega = \omega_k = \frac{2\pi k}{N} \quad k = 0 \text{ to } N-1$$

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & 0 \le \omega \le 0.5\pi \\ e^{-j\alpha\omega} & 1.5\pi \le \omega \le 2\pi \\ 0 & 0.5\pi \le \omega < 1.5\pi \end{cases}$$

Design a linear phase FIR LPF with cut-off frequency of 0.5π rad/sample by taking 11 samples of ideal frequency response.

$\alpha = \frac{N-1}{2}$, $N = 11$

$\alpha = \frac{11-1}{2} = 5$

$\omega_k = \frac{2\pi k}{11}$ $k = 0$ to 10.

distance / difference / sample } ω_k

when $k=0$ $\omega_k =$

$k=0$, $\omega_k = \frac{2\pi 0}{11} = 0$

$k=1$ $\frac{2\pi \times 1}{11} = 0.18\pi$ ✓

$k=2$. ~~$\frac{2\pi 2}{11} = 0.36\pi$~~ $\omega_k = \frac{2\pi \times 2}{11} = 0.36\pi$ ✓

$k=3$ $\omega_k = \frac{2\pi \times 3}{11} = 0.55\pi$

$k=4$ $\omega_k = \frac{2\pi \times 4}{11} = 0.73\pi$

$k=5$ $\omega_k = \frac{2\pi \times 5}{11} = 0.91\pi$

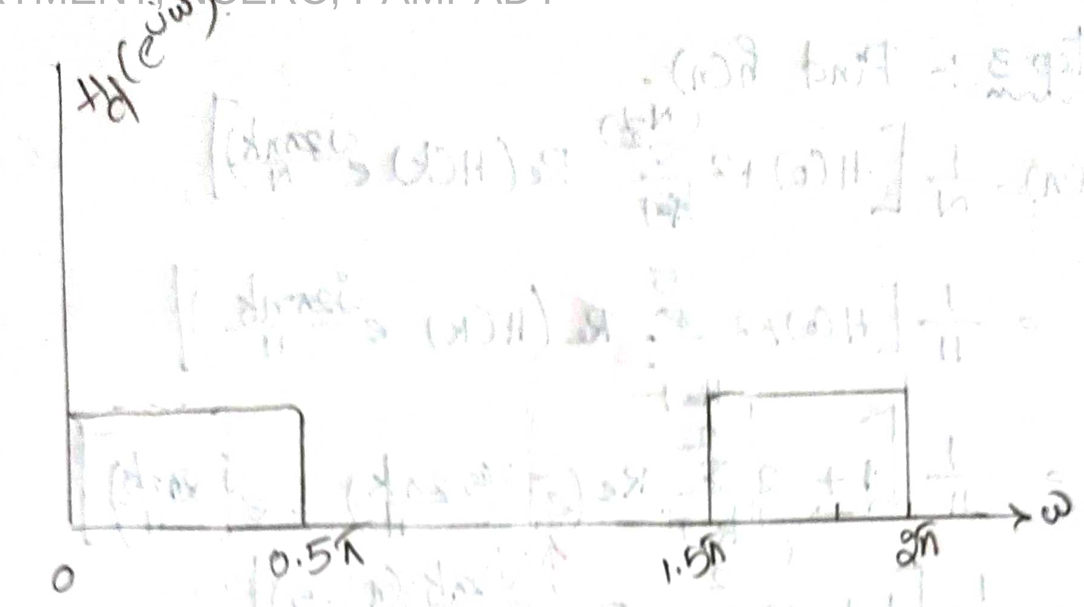
$k=6$ ω_k $\frac{2\pi \times 6}{11} = 1.09\pi$

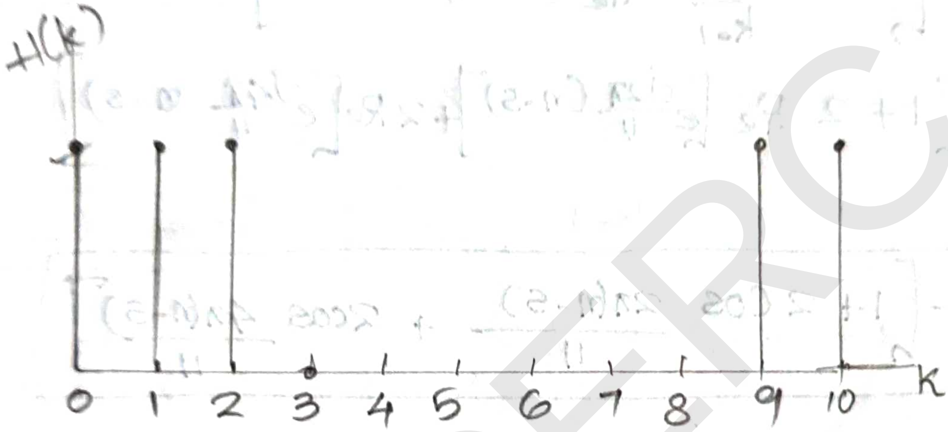
$k=7$ $\omega_k = \frac{2\pi \times 7}{11} = 1.27\pi$

$k=8$ $\omega_k = \frac{2\pi \times 8}{11} = 1.45\pi$

$k=9$ $\omega_k = \frac{2\pi \times 9}{11} = 1.64\pi$ ✓

$k=10$ $\omega_k = \frac{2\pi \times 10}{21} = 1.82\pi$ ✓





k = 0 to 2 samples lie in the range $0 \le \omega \le 0.5\pi$

k = 3 to 8 samples lie in the range $0.5\pi < \omega < 1.5\pi$

k = 9 to 10 samples lie in the range $1.5\pi \le \omega \le 2\pi$.

$$H(k) = H_d(e^{j\omega})\Big|_{\omega=\omega_k} = e^{-j\alpha\omega_k}.$$

$$H(k) = \begin{cases} e^{-j5\left(\frac{2\pi k}{11}\right)} & \text{for } k = 0,1,2 \\ 0 & \text{for } k = 3 \text{ to } 8 \\ e^{-j5\times\frac{2\pi k}{11}} & \text{for } k = 9,10 \end{cases}$$

Step 3 :- Find $h(n)$.

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\left(\frac{N-1}{2}\right)} Re\left(H(k)\, e^{\frac{j2\pi nk}{N}}\right)\right]$$

$$= \frac{1}{11}\left[H(0) + 2\sum_{k=1}^{5} Re\left(H(k)\, e^{\frac{j2\pi nk}{11}}\right)\right]$$

$$= \frac{1}{11}\left[1 + 2\sum_{k=1}^{2} Re\left(e^{-j\left(5\frac{2\pi k}{11}\right)} \cdot e^{j\frac{2\pi nk}{11}}\right)\right]$$

$$= \frac{1}{11}\left[1 + 2\sum_{k=1}^{2} Re\left(e^{\frac{j2\pi k}{11}}\right)^{(n-5)}\right]$$

$$= \frac{1}{11}\left[1 + 2\,Re\left[e^{\frac{j2\pi}{11}(n-5)}\right]_{k=1} + 2\,Re\left[e^{\frac{j4\pi}{11}(n-5)}\right]_{k=2}\right]$$

$$\boxed{h(n) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(n-5)}{11} + 2\cos\frac{4\pi(n-5)}{11}\right]}$$

$e^{j\theta} = \cos\theta + j\sin\theta$

$Re(e^{j\theta}) = \cos\theta$

Calculate $h(n)$ for $n=0$ to 10

Use symmetric condⁿ.

$h(n) = h(N-1-n)$ with Centre of Symmetry at $\left(\frac{N-1}{2}\right)$

So calculate $h(n)$ for $n=0$ to 5.

When $n=0$,

$$h(0) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(0-5)}{11} + 2\cos\frac{4\pi(0-5)}{11}\right]$$

$$\boxed{h(0) = 0.0694}$$

$n=1$

$$h(1) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(1-5)}{11} + 2\cos\frac{4\pi(1-5)}{11}\right]$$

$$\boxed{h(1) = -0.054}$$

$$n=2,\quad h(2) = \frac{1}{11}\left[1 + 2\cos 2\pi\left(\frac{2-5}{11}\right) + 2\cos 4\pi\left(\frac{2-5}{11}\right)\right]$$

$$\boxed{h(2) = -0.1094}$$

$$n=3,\quad h(3) = \frac{1}{11}\left[1 + 2\cos 2\pi\frac{(3-5)}{11} + 2\cos 4\pi\frac{(3-5)}{11}\right]$$

$$\boxed{h(3) = 0.0474}$$

$$h(4) = \frac{1}{11}\left[1 + 2\cos 2\pi\frac{(4-5)}{11} + 2\cos 4\pi\frac{(4-5)}{11}\right]$$

$$\boxed{h(4) = 0.3194}$$

$$h(5) = \frac{1}{11}\left[1 + 2\cos 2\pi\left(\frac{5-5}{11}\right) + 2\cos\frac{4\pi(5-5)}{11}\right]$$

$$\boxed{h(5) = 0.4545}$$

Cond^n. When $n=6$ $h(6) = h(11-1-6) = h(4)$ → $h(N-1-n)$.

$h(6) = h(4) = 0.3194$.

$n=7$ $h(7) = h(11-1-7) = h(3)$

$h(7) = h(3) = 0.0474$

$n=8$ $h(8) = h(11-1-8) = h(2)$

$h(8) = h(2) = -0.1094$.

$n=9$ $h(9) = h(11-1-9) = h(1)$ $h(9) = h(1) = -0.054$

$n=10$ $h(10) = h(11-1-10) = h(0)$ $h(10) = h(0) = 0.0694$.

Step 4 :- Find Transfer function

$$H(z) = \sum_{n=0}^{10} h(n) z^{-n}$$

$$= h(0)z^{-0} + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8} + h(9)z^{-9} + h(10)z^{-10}.$$

Using symmetry Condn.

$h(n) = h(N-1-n)$.

$h(0) = h(10)$, $h(1) = h(9)$

$h(2) = h(8)$, $h(3) = h(7)$

$h(4) = h(6)$.

$$= h(0)[1+z^{-10}] + h(1)[z^{-1}+z^{-9}] + h(2)[z^{-2}+z^{-8}] + h(3)[z^{-3}+z^{-7}] + h(4)[z^{-4}+z^{-6}] + h(5)z^{-5}$$

$$= 0.0694[1+z^{-10}] - 0.054[z^{-1}+z^{-9}] - 0.1094[z^{-2}+z^{-8}] + 0.0474(z^{-3}+z^{-7}) + 0.3194[z^{-4}+z^{-6}] + 0.4545z^{-5}$$

Qn. Design an FIR filter approximating the ideal frequency response.

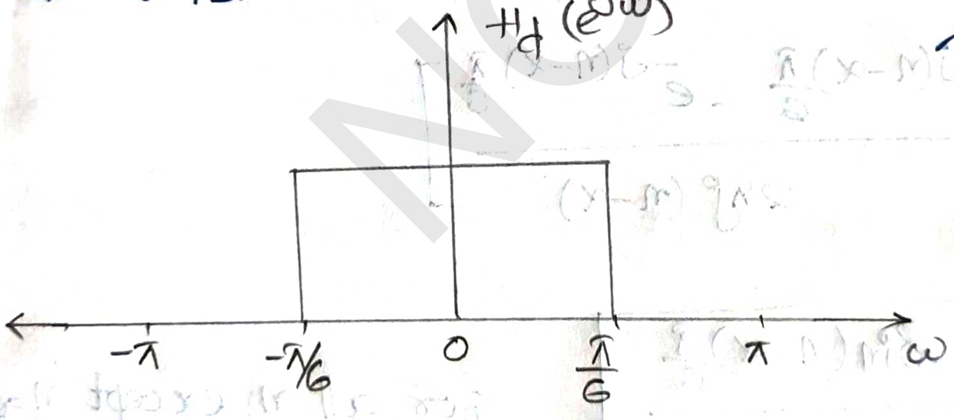
$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & \text{for } |\omega| \le \frac{\pi}{6} \\ 0 & \text{for } \frac{\pi}{6} \le |\omega| \le \pi \end{cases}$$

Use Hamming Window. Determine the filter Coefficients for N=13.

Soln. Given :-
↳ Hamming Window
↳ α ⇒ present, $h(n) = h(N-1-n)$.
↳ N = 13.

Step 1: Plot the desired freq. response.
↳ From the Condition ⇒ LPF.
↳ Hamming Window.
↳ N=13.

From Qn. $\alpha \neq 0$.
$h(n) = h(N-1-n)$
Symmetric Condn.



Step 2:- Find $h_d(n)$ desired impulse response.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/6}^{\pi/6} e^{-j\alpha\omega} \cdot e^{j\omega n}\, d\omega$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\frac{\pi}{6}}^{\pi/6} e^{j(-\alpha+n)\omega}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j(-\alpha+n)\omega}}{j(-\alpha+n)}\right]_{-\frac{\pi}{6}}^{\pi/6}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j(-\alpha+n)\frac{\pi}{6}} - e^{j(-\alpha+n)(-\frac{\pi}{6})}}{j(-\alpha+n)}\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j(n-\alpha)\frac{\pi}{6}} - e^{-j(n-\alpha)\frac{\pi}{6}}}{j(n-\alpha)}\right]$$

$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$$

$$h_d(n) = \left[\frac{e^{j(n-\alpha)\frac{\pi}{6}} - e^{-j(n-\alpha)\frac{\pi}{6}}}{2\pi j\,(n-\alpha)}\right]$$

$$\boxed{h_d(n) = \frac{\sin(n-\alpha)\frac{\pi}{6}}{\pi(n-\alpha)}}$$

for all n, except $n=\alpha$

$n=0$ $\quad h_d(n) = \lim_{n\to\frac{N-1}{2}} h_d(n)$.

$\alpha \neq 0$

$n \to 0$ ✗

$n \to \frac{N-1}{2}$

$\alpha = \frac{N-1}{2} = \frac{13-1}{2} = 6$

$N=13$

$n \to \frac{13-1}{2}$

$\therefore n = \alpha = 6$.

$n \to 6$

$$h_d(6) = \lim_{n \to \frac{N-1}{2}} \frac{\sin(n-\alpha)\frac{\pi}{6}}{\pi(n-\alpha)}.$$

$$= \frac{1}{\pi} \lim_{n \to 6} \left(\frac{\sin(n-6)\frac{\pi}{6}}{(n-6)} \right) \quad (\alpha=6).$$

$$= \frac{1}{\pi} \lim \Rightarrow \frac{1}{\pi} \times \frac{\pi}{6} = \frac{1}{6}$$

$$\boxed{h_d(6) = \frac{1}{6}} \quad \text{for } n=6.$$

Step 3: Hamming Window ⇒ Qn.

Multiply $h_d(n)$ with window sequence $h(n) = h_d(n) W_H(n)$ for $n=0$ to $N-1$.

$$h(n) = h_d(n) \times W_H(n).$$

↳ Use Symmetric Condition $h(n) = h(N-1-n)$.

↳ Here $N=13 \Rightarrow N-1 = 12$.

$$W_H(N) = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right) & \text{for } n=0 \text{ to } N-1 \\ 0 & \text{, otherwise} \end{cases}$$

$$h(0) = h_d(0) \times W_H(0).$$

$$h(0) = \frac{\sin(0-6)\frac{\pi}{6}}{\pi(0-6)} \times \left(0.54 - 0.46 \cos\left(\frac{2\pi \times 0}{12}\right)\right)$$

$$= 0 \times 0.08 = 0$$

$$\boxed{h(0) = 0}$$

$$h(1) = \frac{\sin(1-6)\frac{\pi}{6}}{\pi(1-6)} \times 0.54 - 0.46\left(\cos\frac{2\pi}{12}\right)$$

$$= 0.0318 \times 0.142 = 4.5156\times10^{-3}$$

$$\boxed{h(1) = 0.0045}$$

$$h(2) = \frac{\sin(2-6)\frac{\pi}{6}}{\pi(2-6)} \times 0.54 - 0.46 \cos\left(\frac{2\pi\times2}{12}\right)$$

$$= 0.0689 \times 0.31 = 0.02136$$

$$\boxed{h(2) = 0.02136}$$

$$h(3) = \frac{\sin(3-6)\frac{\pi}{6}}{\pi(3-6)} \times 0.54 - 0.46 \cos\left(\frac{2\pi\times3}{12}\right)$$

$$= 0.106 \times 0.54 = 0.0572$$

$$\boxed{h(3) = 0.0572}$$

$$h(4) = h_d(4) \times w_H(4)$$

$$= \frac{\sin(4-6)\pi/6}{\pi(4-6)} \times 0.54 - 0.46 \cos\left(\frac{2\pi\times4}{12}\right)$$

$$h(4) = 0.1378 \times 0.77 = 0.1061$$

$$\boxed{h(4) = 0.1061}$$

$$h(5) = \frac{\sin(5-6)\pi/6}{\pi(5-6)} \times 0.54 - 0.46 \cos\left(\frac{2\pi\times5}{12}\right)$$

$$= 0.159 \times 0.94 = 0.149$$

$$\boxed{h(5) = 0.149}$$

$$h(6) = \frac{\sin(6-6)\pi/6}{\pi(6-6)} \times 0.54 - 0.46 \cos\left(\frac{2\pi \times 6}{12}\right)$$

$$= 0.167 \times 1 = 0.167$$

$$\boxed{h(6) = 0.167}$$

Symmetric Condⁿ. $h(n) = h(N-1-n)$.

$h(7) = h(13-1-7) = h(5)$.

$$\boxed{h(7) = h(5) = 0.149}$$

$h(8) = h(13-1-8) = h(4)$

$$\boxed{h(8) = h(4) = 0.1061}$$

$h(9) = h(13-1-9) = h(3)$

$$\boxed{h(9) = h(3) = 0.0572}$$

$h(10) = h(13-1-10) = h(2)$

$$\boxed{h(10) = h(2) = 0.0213}$$

$h(11) = h(13-1-11) = h(1)$

$$\boxed{h(11) = h(1) = 0.0045}$$

$h(12) = h(13-1-12) = h(0)$.

$$\boxed{h(12) = h(0) = 0}$$

Step 4:- To find the transfer fn $H(z)$

$H(z) =$

Qn Using a rectangular window technique design a LPF with passband gain of unity, Cut-off frequency of 1000 Hz and working at a sampling frequency of 5 kHz. The length of impulse response should be 7.

Soln. Given

- LPF
- Passband gain of Unity ⇒ 1
- $F_c = 1000 Hz$
- Sampling freq. $F_s = 5 kHz$
- Window ⇒ Rectangular window.

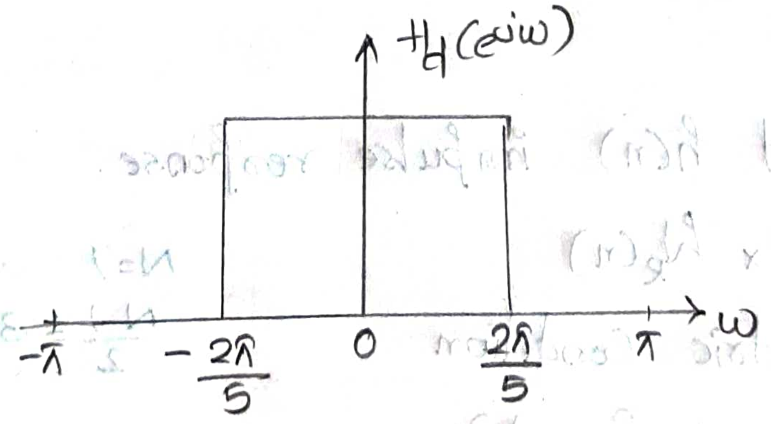
freq ⇒ Hz to π. Convert

$$\omega_c = 2\pi f = \frac{2\pi F_c}{F_s} = \frac{2\pi \times 1000}{5\times 10^3}$$

$$\boxed{\omega_c = \frac{2\pi}{5}} \text{ rad.}$$

Frequency response.

$$H_d(e^{j\omega}) = \begin{cases} 1 & |\omega_c| \le \frac{2\pi}{5} \\ 0 & \text{Otherwise} \end{cases}$$



Step 2: To find $h_d(n)$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\frac{2\pi}{5}}^{\frac{2\pi}{5}} 1.e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\frac{2\pi}{5}}^{\frac{2\pi}{5}} = \frac{1}{2\pi}\left[\frac{e^{j\frac{2\pi}{5}n} - e^{-j\frac{2\pi}{5}n}}{jn}\right]$$

$$\boxed{h_d(n) = \frac{\sin\left(\frac{2\pi n}{5}\right)}{\pi n}}$$ for all n, except $n=0$

For $n=0$,

$$h_d(n) = \lim_{n\to 0} \frac{\sin\left(\frac{2\pi n}{5}\right)}{\pi n}$$

$$h_d(0) = \frac{1}{\pi}\lim_{n\to 0}\frac{\sin\left(\frac{2\pi n}{5}\right)}{\pi n} = \frac{1}{\pi}\times\frac{2\pi}{5}$$

$$\boxed{h_d(0) = \frac{2}{5}}$$

Step 3: To Find $h(n)$ impulse response.

$$h(n) = h_d(n) \times W_R(n)$$

$N=7$

$\frac{N-1}{2} = 3.$

Using Symmetric Condition

~~$h(n) = h(N-1-n)$~~ ~~$h(-n)$~~).

$$h(n) = h(-n).$$

$$W_R(n) = \begin{cases} 1 & \text{for } -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

$$h(0) = h_d(0) \times W_R(0)$$

$$= \frac{2}{5} \times 1 = \frac{2}{5}$$

$$\boxed{h(0) = \frac{2}{5}}$$

$$h(1) = \frac{\sin\frac{2\pi}{5}}{\pi} \times 1 = 0.3027 \quad \boxed{h(1) = 0.3027} = h(-1)$$

$$h(2) = h(-2) = \frac{\sin\frac{2\pi \times 2}{5}}{2\pi} = 0.0935$$

$$\boxed{h(2) = h(-2) = 0.0935}$$

$$h(3) = h(-3) = \frac{\sin\frac{2\pi \times 3}{5}}{3\pi} = -0.0624$$

$$\boxed{h(3) = h(-3) = -0.0624.}$$

<u>Step 4</u>:- Find transfer fn $H(z)$

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n) z^{-n}$$

$$= z^{-3}\left[h(-3)z^3 + h(-2)z^2 + h(-1)z^1 + h(0)z^0 + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}\right]$$

$$= h(-3)z^0 + h(-2)z^{-1} + h(-1)z^{-2} + h(0)z^{-3} + h(1)z^{-4} + h(2)z^{-5} + h(3)z^{-6}.$$

DESIGN OF IIR FILTER

- IIR → Infinite impulse response filter.
- One of the most widely used complex signal processing operations → its operation is filtering.
- Main objective is to alter the spectrum according to some given specifications.

  Eg:- LPF ⇒ Low Signals } Specifications.
  HPF ⇒ High Signals }

- The system implementing this operation is called a filter.
- The discrete-time system for the treatment of discrete-time signal is called a digital filter.

Classification of Digital filter

- * Classical filters. → LPF, HPF, BPF, BSF → Based on Specifications
- * Modern filters

According to the length of their impulse response sequences, the digital filters are usually classified as

- Finite impulse response (FIR) Filters.

$$H(z) = \sum_{n=0}^{M-1} h(n) z^{-n}$$

- Infinite Impulse response (IIR) Filters

$$H(z) = \frac{\sum_{r=0}^{M} b_r z^{-r}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

- Important step in the development of a digital filter is the determination of a ~~real~~ realizable transfer fn. $H(z)$ approximating the gn. frequency response specifications.
- The process of deriving the transfer function $H(z)$ is called digital filter design.
- A transition band is specified b/w the passband ~~b/w the~~ and the stopband to permit the magnitude to drop off smoothly.
- The magnitude response specifications of a digital filter in passband and in the stopband are given with some acceptable tolerance.

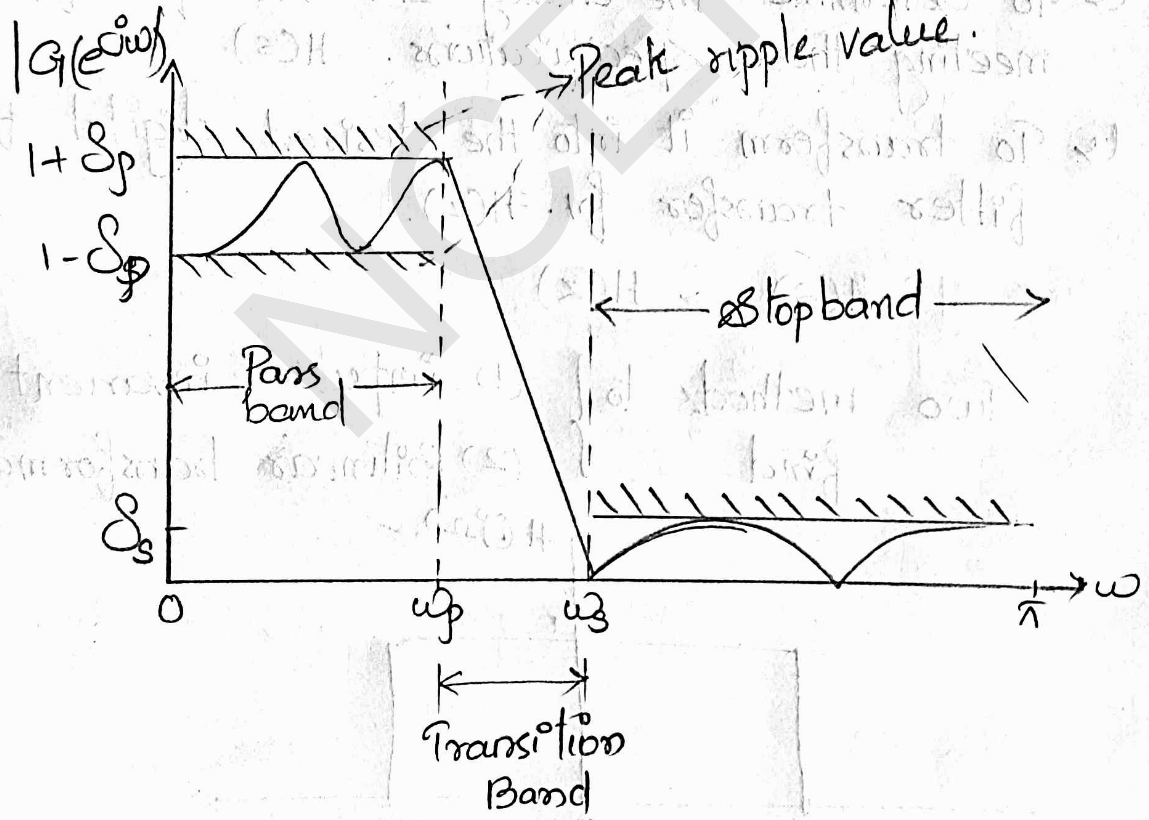


Fig:- Magnitude specifications for a digital LPF

δ_p – Ripple value.

↳ Digital filter specifications are often given in terms of loss fn.

$$A(\omega) = -20 \log_{10} |G(e^{j\omega})| \text{ in dB.}$$

Peak pass band ripple

$$\alpha_p = -20 \log_{10}(1-\delta_p) \text{ dB.}$$

Minimum stop band attenuation

$$\alpha_s = -20 \log_{10}(\delta_s) \text{ dB}$$

Concepts

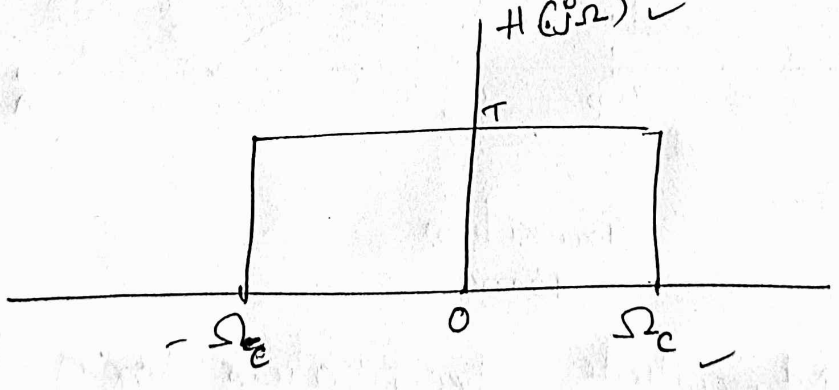
↳ To convert the digital filter specifications into analog lowpass prototype filter specifications.

↳ To determine the analog LPF transfer fn. meeting these specifications. H(s)

↳ To transform it into the desired digital filter transfer fn. H(z).

$$H(s) \longrightarrow H(z)$$

Two methods to find } (1) impulse invarient
(2) Bilinear transformation.



Magnitude response of a ideal LPF

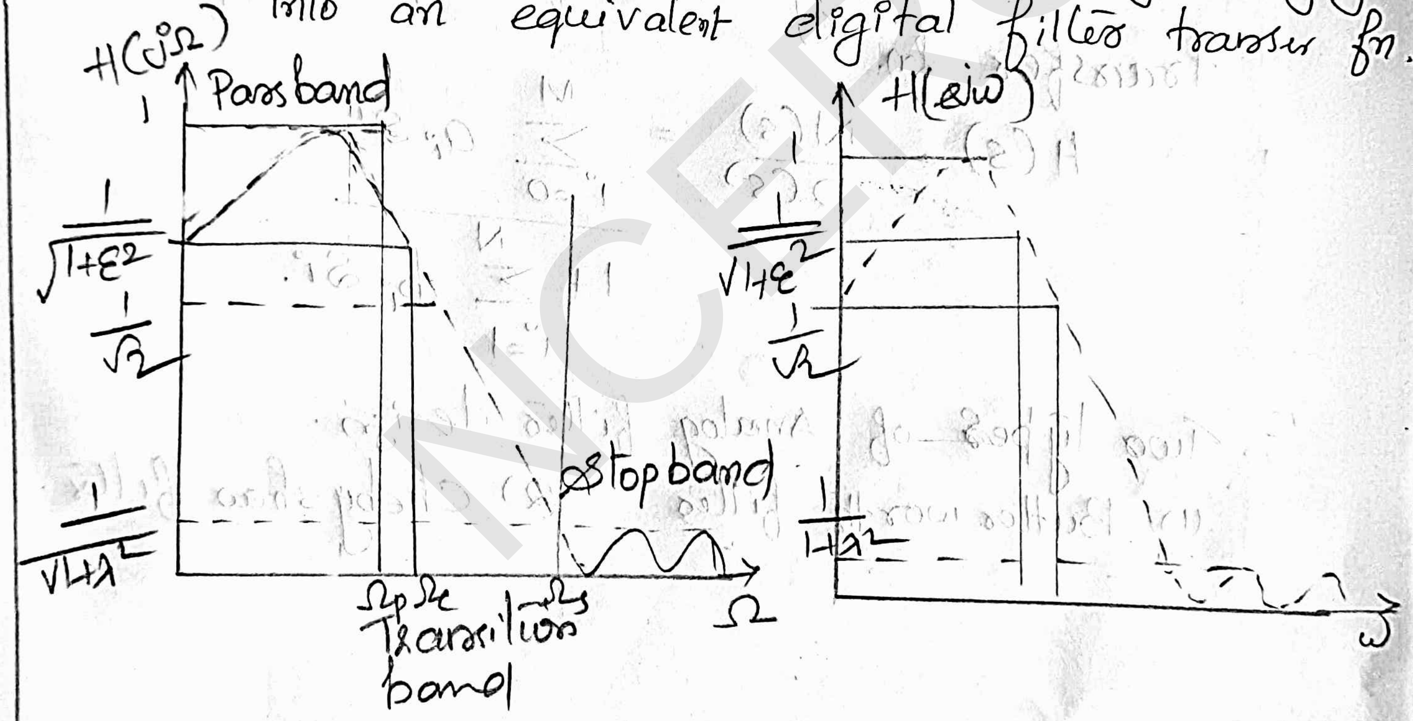
DESIGN OF IIR FILTER

- Indirect method.
- First design an analog prototype ~~after~~ filter and then transform the prototype to a digital filter.

Step 1 ⇒ Map the desired digital filter specifications into those for an equivalent analog filter

Step 2 ⇒ Derive the analog transfer fn. for the analog prototype

Step 3 ⇒ Transform the transfer of analog prototype into an equivalent digital filter transfer fn.



Magnitude Respons

Analogy LPF

Digital LPF

110p

$$\varepsilon = \frac{2\sqrt{\delta_p}}{1-\delta_p}$$

$$\lambda = \frac{(1+\delta_p)^2 - \delta_s^2}{\delta_s}$$

Analog Low Pass filter Design

The most general form of analog filter transfer fn

$$H(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=0}^{M} a_i s^i}{1 + \sum_{i=1}^{N} b_i s^i}.$$

Two types of Analog filter design.

(1) Butterworth filter (2) Chebyshew filter.

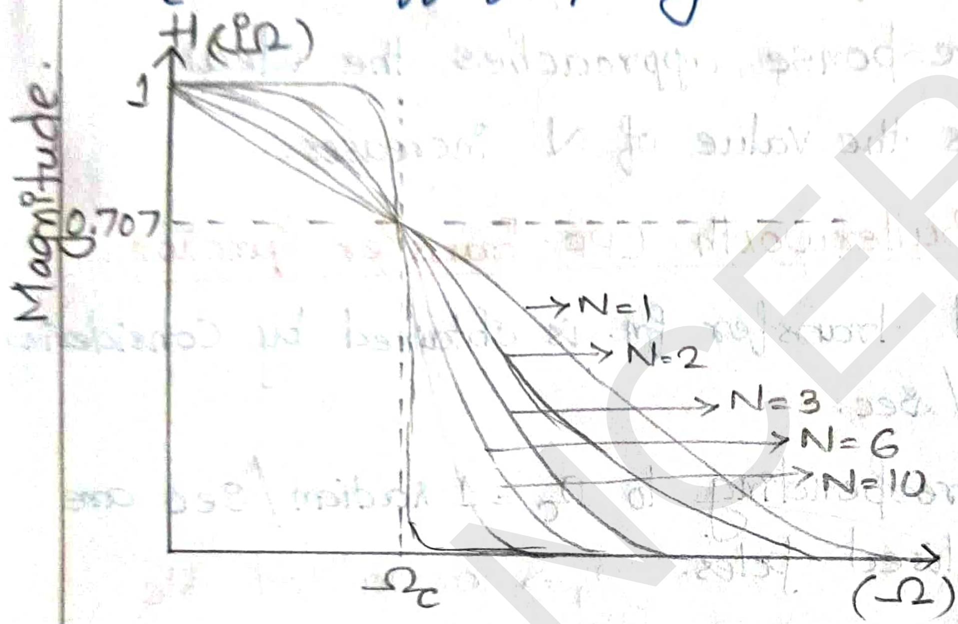
Analog Lowpass Butterworth filter

Magnitude fn of Butterworth low pass filter is given by.

$$|H(j\Omega)| = \frac{1}{\left(1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right)^{1/2}} \qquad N=1,2,3,\ldots$$

N = Order of filter.

Ω_c = Cut-off frequency



(Ω) radian/Sec

Frequency.

Fig:- Lowpass Butterworth magnitude response

↳ change in characteristics of filter depends on the order of filter.

↳ Order of the filter increases it approximates to the ideal filter response.

N = 10, its near to Ω_c

↳ For a stable filter all poles should lie in left half of S-plane.

The eqn. for poles of Butterworth filter is gn. by

Imp: $s_k = e^{j\phi_k}$ where $\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$ $\quad k=1,2,3\ldots N$

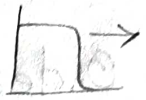
Properties of Butterworth Filter

(i) Butterworth filters are all pole designs.

(ii) At cut-off frequency Ω_c the magnitude of normalised Butterworth filter is $\frac{1}{\sqrt{2}}$.

(iii) The filter Order N Completely satisfies the filter.

(iv) The magnitude is maximally flat at the origin.


→ No ripple

(v) The magnitude is monotomically decreasing fn of Ω. Frequency ↑se → Mag. ↓se

(vi) Magnitude response approaches the ideal response as the value of N increases.

Normalised Butterworth LPF Transfer function.

↳ The normalised transfer fn. is obtained by Considering $\Omega_c = 1$ radian/sec.

↳ The poles Corresponding to $\Omega_c = 1$ radian/sec are called normalised poles. $S_n \Rightarrow$ normalised Pole

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N Order	Normalised transfer function $H(S_n)$
1	$\frac{1}{S_n+1}$
2	$\frac{1}{S_n^2+1.414S_n+1}$
3	$\frac{1}{(S_n+1)(S_n^2+S_n+1)}$
4.	$\frac{1}{(S_n^2+0.765S_n+1)(S_n^2+1.848S_n+1)}$
5	$\frac{1}{(S_n+1)(S_n^2+0.618S_n+1)(S_n^2+1.618S_n+1)}$

Qn. Determine the poles of lowpass Butterworth filter for N=2. Determine the normalised transfer fn.

Soln

Poles of LP Butterworth filter,

$S_n = e^{j\phi_k}$ $\quad \phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \quad k = 1, 2, 3 \cdots N$

Here N=2 $\quad \therefore$ k=1, 2.

$\phi_1 = \frac{\pi}{2} + \frac{(2-1)\pi}{2\times 2} = \frac{\pi}{2} + \frac{\pi}{4} = \frac{2\pi + \pi}{4} = \frac{3\pi}{4}$

$\boxed{\phi_1 = \frac{3\pi}{4}}$ ✓

$\phi_2 = \frac{\pi}{2} + \frac{(2\times 2 - 1)\pi}{2\times 2} = \frac{\pi}{2} + \frac{(4-1)\pi}{4} = \frac{\pi}{2} + \frac{3\pi}{4} = \frac{5\pi}{4}$

$\boxed{\phi_2 = \frac{5\pi}{4}}$ ✓

For k=1, $S_n = e^{j\phi_1} \Rightarrow e^{j\frac{3\pi}{4}} = \cos\left(\frac{3\pi}{4}\right) + j\sin\frac{3\pi}{4}$

Pole, $P_1 = -0.707 + j0.707$.

For k=2, $S_n = e^{j\phi_2} \Rightarrow e^{j\frac{5\pi}{4}} = \cos\left(\frac{5\pi}{4}\right) + j\sin\left(\frac{5\pi}{4}\right)$

Pole, $P_2 = -0.707 - j0.707$ (Calculator in radian mode)

Normalised Transfer fn,

$$H(S_n) = \frac{1}{(S_n - P_1)(S_n - P_2)}$$

$$= \frac{1}{(S_n - (-0.707 + j0.707))(S_n - (-0.707 - j0.707))}$$

$$\frac{1}{(S_n+0.707-j0.707)(S_n+0.707+j0.707)}$$

$$= \frac{1}{(S_n+0.707)^2-(j0.707)^2} \quad (a^2-b^2)$$

$j^2=-1$

$$= \frac{1}{(S_n+0.707)^2+(0.707)^2}$$

$(a+b)^2 = a^2+b^2+2ab$

$$= \frac{1}{S_n^2+2\times 0.707 \times S_n+(0.707)^2+(0.707)^2}$$

$$\boxed{H_2(S_n) = \frac{1}{S_n^2+1.414S_n+1}}$$

Derive

Order of Butterworth Filter.

digital: $\omega \Rightarrow$ radian/sample

analog: $\Omega \Rightarrow$ radian/sec

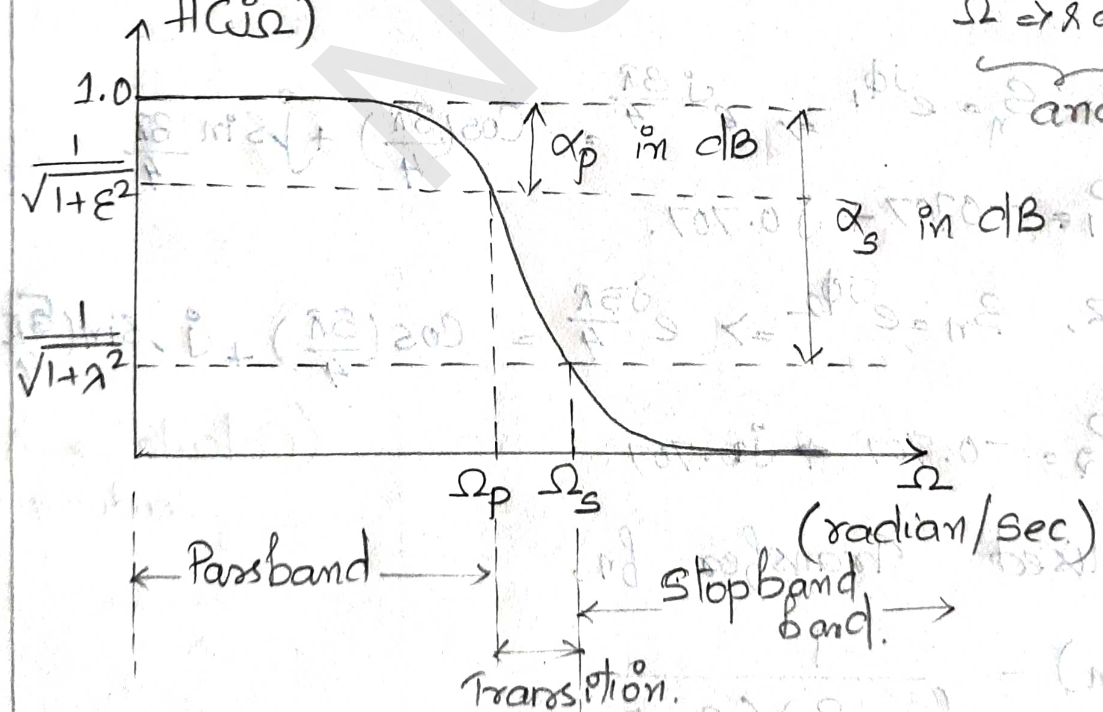


Fig:- Magnitude response of Butterworth filter.

↳ $\alpha_p \Rightarrow$ Max. passband attenuation in +ve dB at passband frequency Ω_p.

↳ $\alpha_s \Rightarrow$ minimum stopband attenuation in +ve dB at stopband frequency Ω_s

↳ The magnitude fn. can be written as

$$|H(j\Omega)| = \frac{1}{\left(1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}\right)^{1/2}} \quad \text{——(1)}$$

N → order of filter.

Squaring on both sides.

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}} \quad \text{——(2)}$$

$|H(j\omega)|^2 = 2 \times 10 \log H(j\Omega)$

↳ Taking logarithm on both sides,

$$20 \log |H(j\Omega)| = \underset{0}{\underline{10 \log 1}} - 10 \log \left(1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}\right) \quad \text{——(3)}$$

↳ At $\Omega = \Omega_p$ attenuation $= \alpha_p$

$$\therefore 20 \log |H(j\Omega)| = -\alpha_p \quad \text{——(4)}$$

$$-\alpha_p = -10 \log \left(1+\varepsilon^2\left(\frac{\Omega_p}{\Omega_p}\right)^{2N}\right) \qquad \left(\frac{\Omega_p}{\Omega_p}\right)^{2N} = 1^{2N} = 1$$

Below 1 value need to give (−)

$$\alpha_p = -10 \log (1+\varepsilon^2) \quad \text{——(5)}$$

$$\frac{1}{10}\alpha_p = \log (1+\varepsilon^2)$$

$$0.1\,\alpha_p = \log (1+\varepsilon^2)$$

↳ Taking antilog on both sides.

$$1+\varepsilon^2 = 10^{0.1\alpha_p}$$

$$\boxed{\varepsilon = \left(10^{0.1\alpha_p} - 1\right)^{1/2}} \quad \text{——(6)}$$

At $\Omega = \Omega_s$ minimum stopband attenuation $\Rightarrow \alpha_s$

$\therefore$ Eqn (3) becomes.

$$20 \log |H(j\Omega_s)| = 10 \log 1 - 10 \log \left[1 + \varepsilon^2 \left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right] \quad \text{—(7)}.$$

$$20 \log |H(j\Omega_s)| = -\alpha_s$$

$$-\alpha_s = -10 \log \left[1 + \varepsilon^2 \left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right]$$

Below 1 Value need to give (−)

$$\frac{1}{10}\alpha_s = \log \left[1 + \varepsilon^2 \left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right]$$

$$0.1\alpha_s = \log \left[1 + \varepsilon^2 \left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right]$$

Taking antilog on both sides,

$$1 + \varepsilon^2 \left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = 10^{0.1\alpha_s}$$

$$\varepsilon^2 \left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = 10^{0.1\alpha_s} - 1$$

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s} - 1}{\varepsilon^2} \quad \text{—(8)}$$

Sub. eqn (6) in (8)

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1} \quad \text{—(9)}$$

Taking logarithm for eqn (10), expression obtained for the order of the filter

Very imp. eqn.

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log \frac{\Omega_s}{\Omega_p}} \quad \text{— (10)}$$

Since this expression normally does not result in an integer value, roundoff N to the next higher integer,

$$N \geq \frac{\log(\lambda/\varepsilon)}{\log(\Omega_s/\Omega_p)} \quad \text{— (11)}$$

where, $\varepsilon = \sqrt{10^{0.1\alpha_p}-1}$

$\lambda = \sqrt{10^{0.1\alpha_s}-1}$

Qn Given the specification $\alpha_p = 1\,dB$, $\alpha_s = 30\,dB$, $\Omega_p = 200$ rad/sec, $\Omega_s = 600$ rad/sec. Determine the order of filter.

Soln.

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log \frac{\Omega_s}{\Omega_p}}$$

$$= \frac{\log \sqrt{\frac{10^{0.1 \times 30} - 1}{10^{0.1 \times 1} - 1}}}{\log \left(\frac{600}{200}\right)} = \frac{\log 62.115}{\log 3} = 3.758$$

$N \geq 3.758$ Round-off to next integer.

$\therefore N = 4$

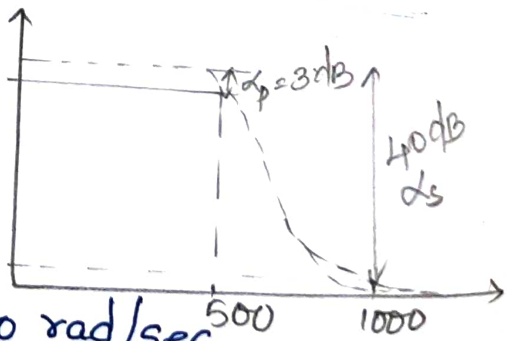
Qn Determine the order and poles of lowpass Butterworth filter that has a 3dB attenuation at 500 Hz and an attenuation of 40 dB at 1000 Hz.

$\alpha_p = 3\,dB$ $\quad \alpha_s = 40\,dB$

$F_p = 500\,Hz \Rightarrow$

$\Omega_p = 2\pi \times 500 = 1000$ rad/sec

$F_s = 1000\,Hz \quad \Omega_s = 2\pi \times 1000 = 2000$ rad/sec



Order $N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log \left(\frac{\Omega_s}{\Omega_p}\right)}$

$$N \geq \frac{\log \sqrt{\frac{10^{\dots}-1}{10^{0.3}-1}}}{\log\left(\frac{2000\pi}{1000\pi}\right)} = 6.6$$

$$\boxed{N = 7}$$

Poles of Butterworth filter given by

$$S_k = \Omega_c \, e^{j\phi_k}$$

$\Omega_c \rightarrow$ Cut off frequency $\Rightarrow$ pass band attenuation at 3dB

$\therefore$ 3B attenuation $\Omega_c = \Omega_p = 1000\pi$.

$$S_k = 1000\pi \, e^{j\phi_k} \qquad k = 1, 2, \dots 7$$

where. $\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1, 2, \dots 7$

Steps to design an Analog Butterworth LPF.

1. Find the order of the filter N.
2. Round off it to next higher integer value.
3. Find transfer fn H(s) for $\Omega_c = 1$ rad/sec for the values of N.
4. Calculate the value of cut-off frequency Ω_c
5. Find transfer function $H_a(s)$ for the above value of Ω_c by substituting $s \to \frac{s}{\Omega_c}$ in H(s)

Qn. Design an analog Butterworth filter that has a -2 dB passband attenuation at a frequency of 20 rad/sec and atleast -10 dB stopband attenuation at 30 rad/sec.

Soln. Given data

$\alpha_p = 2$ dB $\quad \Omega_p = 20$ rad/sec

$\alpha_s = 10$ dB $\quad \Omega_s = 30$ rad/sec

Take modulus of α_p and α_s

Step 1:- Order of filter

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$\Rightarrow N \geq \frac{\log \sqrt{\frac{10^{0.1 \times 10} - 1}{10^{0.1 \times 2} - 1}}}{\log\left(\frac{30}{20}\right)} = 3.37$$

Step 2 :- Rounding off N to next higher integer $\boxed{N=4}$

Step 3:- From table.

Find $H(s)$ for $\Omega_c = 1$ rad/sec

$N = 4$

From table, normalized low pass Butterworth filter for $N = 4$,

$$H(S_n) = \frac{1}{(S_n^2 + 0.765\,S_n + 1)(S_n^2 + 1.848\,S_n + 1)}$$

Step 4:- Find Ω_c

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}}$$

$$\Omega_c = \frac{20}{(10^{0.1\times 2} - 1)^{1/2\times 4}}$$

$$\boxed{\Omega_c = 21.3868}$$

Step 5: Find $H_a(s)$ $\Rightarrow$ $S \rightarrow \frac{S}{\Omega_c}$ $\quad S_n \rightarrow \frac{S}{\Omega_c}$

$$H_a(s) = \frac{1}{\left[\left(\frac{S}{21.3868}\right)^2 + 0.765\times\frac{S}{21.3868} + 1\right]\left[\left(\frac{S}{21.3868}\right)^2 + 1.848\,\frac{S}{21.3868} + 1\right]}$$

$=$

$$= \frac{0.20921\times 10^6}{(S^2 + 16.3686\,S + 457.394)(S^2 + 39.517S + 457.394)}$$

Frequency Transformation in Analog Domain.

Frequency transformations can be used to design lowpass filters with different passband frequencies, highpass filters, bandpass filters and bandstop filters from a normalized lowpass analog filter. ($\Omega_c = 1$ rad/sec).

(a) Lowpass to Lowpass Filter.

Given ⇒ normalized LPF ($\Omega_c = 1$ rad/sec)

↳ Design a LPF with a different cut-off frequency Ω_c (or passband frequency Ω_p).

↳ This can be accomplished by the transformation

$$\boxed{s \rightarrow \frac{s}{\Omega_c}}$$

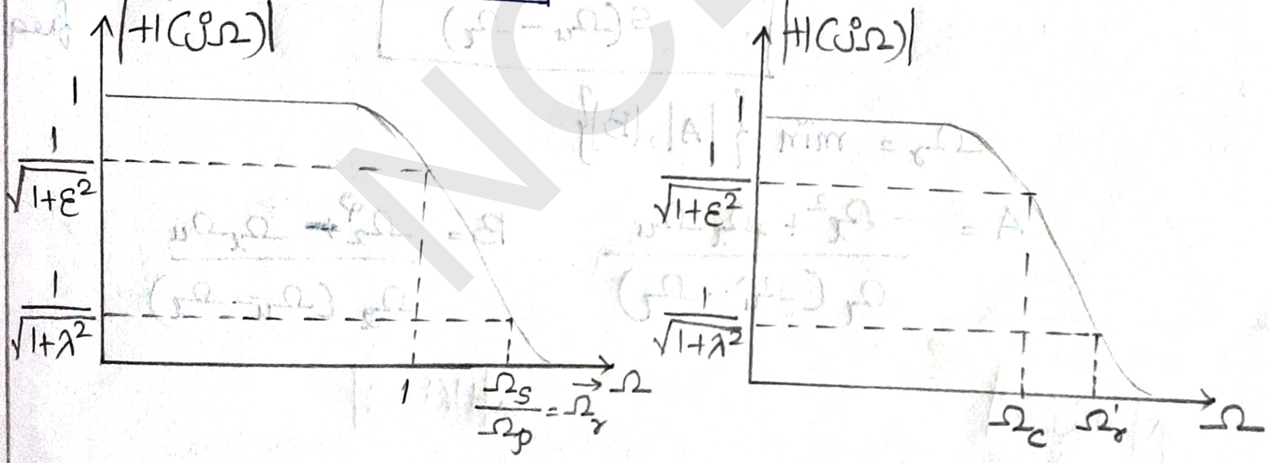


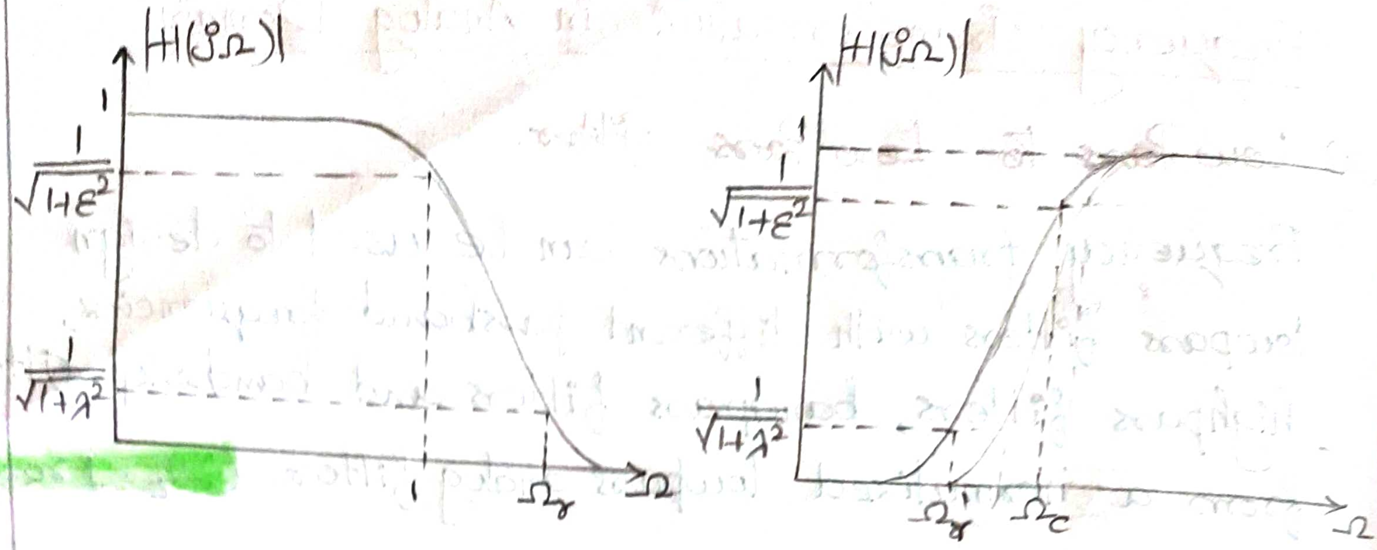
Fig:- Lowpass to Lowpass transformation.

(b) Lowpass to Highpass

Given ⇒ Normalized LPF ($\Omega_c = 1$ rad/sec)

Design ⇒ High pass filter.

Transformation ⇒ $\boxed{s \rightarrow \frac{\Omega_c}{s}}$



Lowpass to Highpass Transformation.

(c) Lowpass to Bandpass.

Given ⇒ normalised LPF ($\Omega_c = 1$ rad/sec)

Design ⇒ Bandpass Filter with cutoff frequencies Ω_l, Ω_u.

$\Omega_l \rightarrow$ lower cut-off freq.

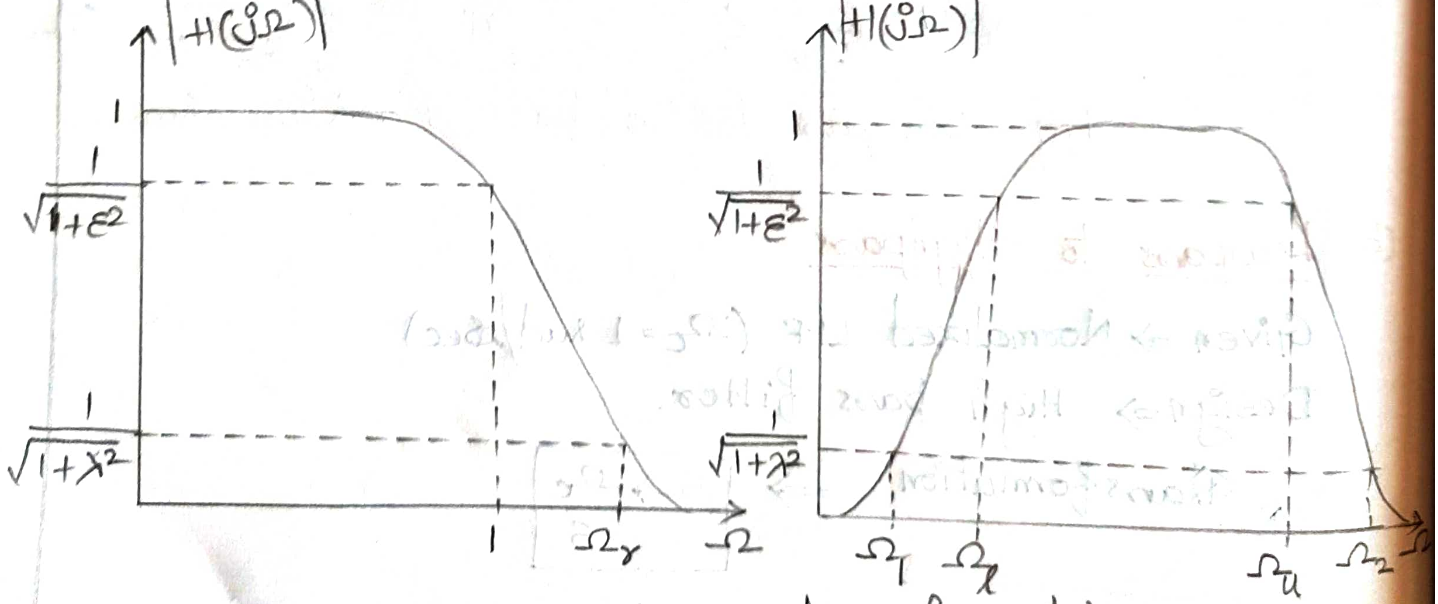
$\Omega_u \rightarrow$ Upper cutoff freq.

Transformation

$$s \rightarrow \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$$

$$\Omega_r = \min\{|A|, |B|\}$$

$$A = \frac{-\Omega_1^2 + \Omega_l \Omega_u}{\Omega_1(\Omega_u - \Omega_l)} \qquad B = \frac{\Omega_2^2 + \Omega_l \Omega_u}{\Omega_2(\Omega_u - \Omega_l)}$$



Lowpass to Bandpass transformation.

(d) Lowpass to Bandstop

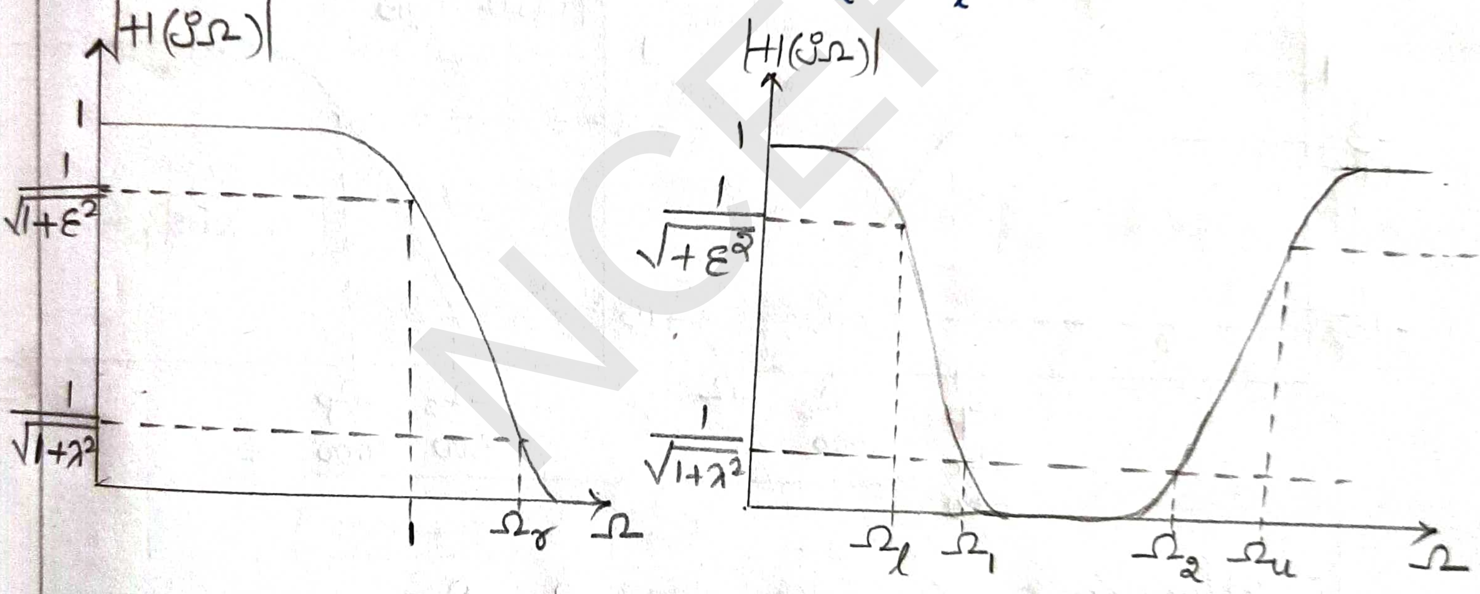
Given ⇒ Normalaized LPF ($\Omega_c = 1$ rad/sec)

Design ⇒ Bandstop filter with cut off frequencies Ω_l and Ω_u

$$s \rightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$$

$$\Omega_r = \min\{|A|, |B|\}$$

$$A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_l \Omega_u} \qquad B = \frac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_l \Omega_u}$$



Lowpass to Bandstop Transformation.

Qn. For the given specifications $\alpha_p = 3\,dB$, $\alpha_s = 15\,dB$, $\Omega_p = 1000$ rad/sec and $\Omega_s = 500$ rad/sec design a highpass filter.

Soln. First design a normalized LPF and then use a transformation to get the transfer fn of a HPF.

For HPF

$\Omega_c = \Omega_p = 1000$ rad/sec

$\Omega_s = 500$ rad/sec

$\alpha_p = 3\,dB$

$\alpha_s = 15\,dB$

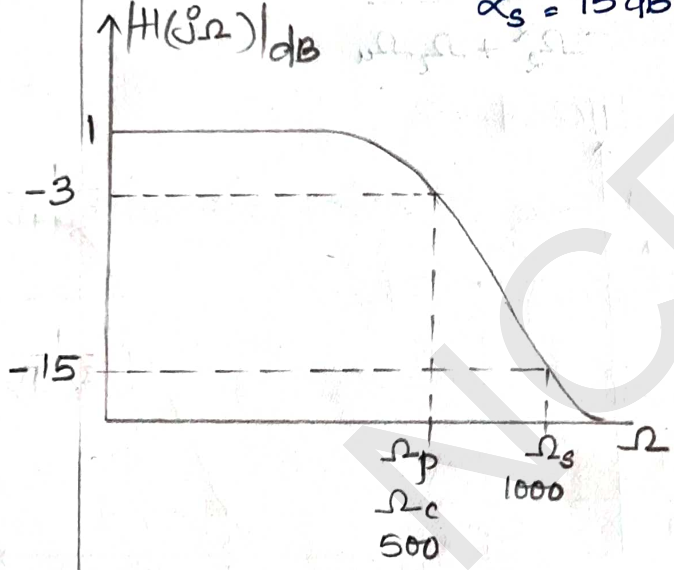
~~For HP LPF~~

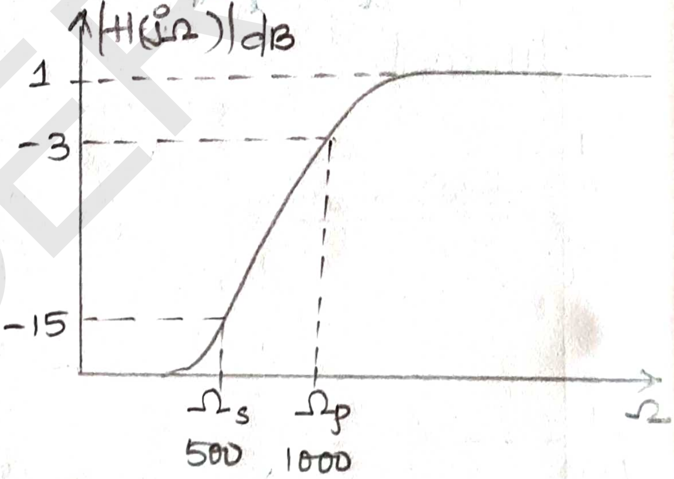
For LPF

{ Ω_s of HPF = Ω_p of LPF }

$\Omega_c = \Omega_p = 500$ rad/sec

$\Omega_s = 1000$ rad/sec





Whenever $\alpha_p = 3\,dB$ then $\Omega_p = \Omega_c$

Step 1 :- Order of filter

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\times 15} - 1}{10^{0.1\times 3} - 1}}}{\log\left(\frac{1000}{500}\right)} = \frac{\log 5.533}{\log 2} = 2.468$$

Step 2 :- Round off to next higher integer.

$$\boxed{N = 3}$$

Step 3 :- Find $H(S_n)$

For $N=3$, $H(S_n) = \dfrac{1}{(S_n+1)(S_n^2+S_n+1)}$

Step 4 :- Transfer function $H_a(s)$

To get HPF having cutoff freq

$\Omega_c = \Omega_p = 1000$ rad/sec

Sub. $S_n \rightarrow \dfrac{\Omega_p}{S} \Rightarrow S_n \rightarrow \dfrac{1000}{S}$

$$H_a(s) = H(S_n)\Big|_{S_n \rightarrow \frac{1000}{S}}$$

$$H_a(s) = \frac{1}{\left(\frac{1000}{S}+1\right)\left(\left(\frac{1000}{S}\right)^2 + \frac{1000}{S}+1\right)}$$

$$= \frac{1}{\left(\frac{1000+S}{S}\right)\left(\frac{1000^2}{S^2} + \frac{1000}{S}+1\right)}$$

$$= \frac{1}{\left(\frac{1000+S}{S}\right)\left(\frac{1000^2 + 1000S + S^2}{S^2}\right)}$$

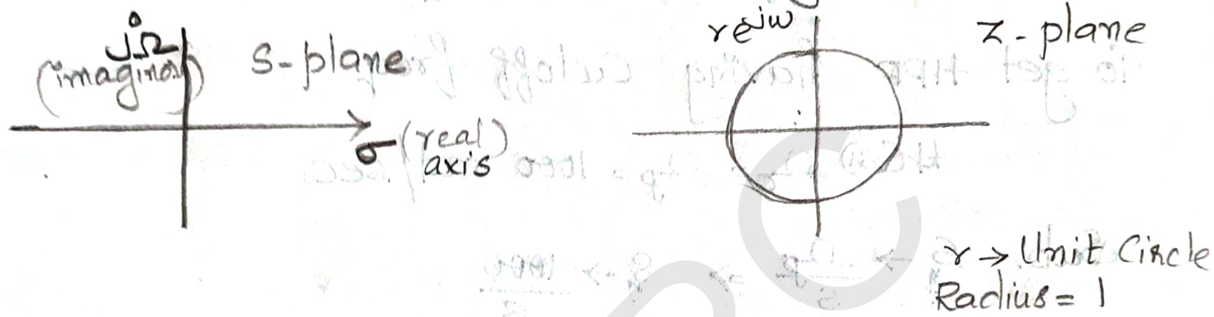
$$H_a(s) = \frac{S^3}{(1000+S)(S^2+1000S+1000^2)}$$

IIR FILTER DESIGN BY IMPULSE INVARIANCE

↳ Indirect method $H(s) \longrightarrow H(z)$

↳ The analog filter can be converted to digital filter by considering the following properties.

(a) The $j\Omega$-axis of s-plane (Analog) should map into (on) the unit circle in the z-plane (digital)



(b) The left half plane of s-plane should map into the inside of the unit circle in the z-plane.

Methods to design digital filter from analog filter.

(1) Impulse invariant transformation
(2) Bilinear transformation.

Impulse Invariance Technique

↳ In Impulse invariance method,

Sampled version of impulse response of analog filter $\Rightarrow$ Unit impulse response $h(n)$ of digital filter.

↳ The z-transform of IIR is given by

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \qquad (1)$$

$$H(z)\Big|_{z=e^{sT}} = \sum_{n=0}^{\infty} h(n) e^{sTn} \qquad (2)$$

Let us consider the mapping of points from s-plane to z-plane given by.

$$z = e^{sT} \longrightarrow \text{mapping eqn.} \quad —(3)$$

↳ Sub. $s = \sigma + j\Omega$ and express the complex variable z in polar form as $z = re^{j\omega}$,

$$\therefore \quad re^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$re^{j\omega} = e^{\sigma T} \cdot e^{j\Omega T}$$

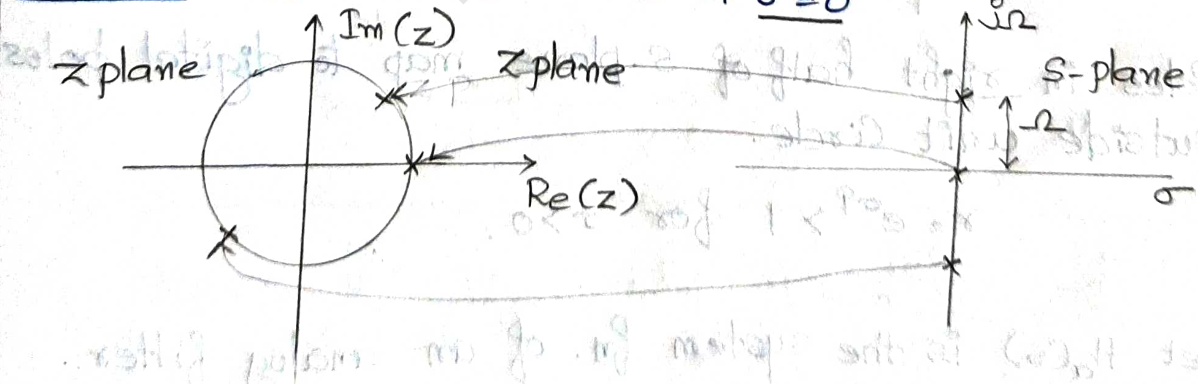
Equating real and complex parts.

$$r = e^{\sigma T} \quad \text{(real)}$$
$$\omega = \Omega T \quad \text{(complex)} \quad —(4)$$

Real part of analog pole = Radius of z-plane pole

Imaginary part of analog pole = Angle of digital pole (z-plane)

<u>Case I</u> : Pole on $j\Omega$ axis $\Rightarrow$ <u>$\sigma = 0$</u>

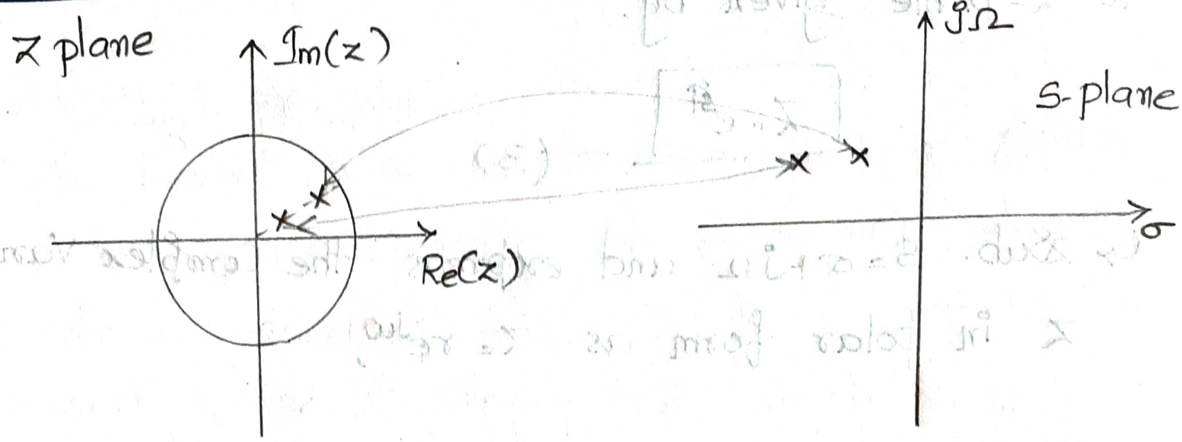


Poles map to z plane at radius r

$$r = e^{\sigma T} = e^{0.T} = 1$$

$\therefore$ Impulse invariant mapping map poles from s-Plane's $j\Omega$ to z-plane's Unit circle.

Case II Poles in left half of s-plane $\sigma < 0$

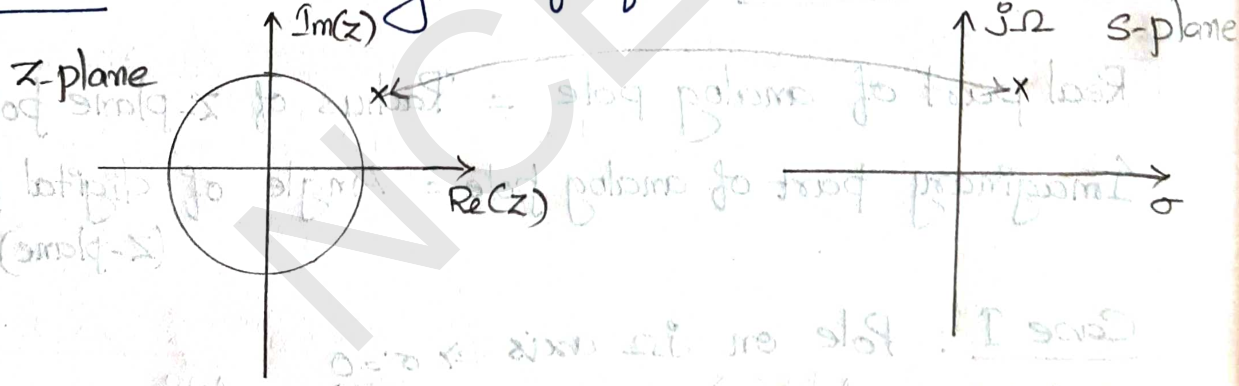


Poles map inside the unit circle

as $r = e^{\sigma T} < 1$ for $\sigma < 0$

∴ All s-plane poles with −ve real parts map to z-plane poles inside unit circle.

Case III Poles in right half of s-plane, $\sigma > 0$



Poles in right half of s-plane map to digital poles outside unit circle.

$$r = e^{\sigma T} > 1 \text{ for } \sigma > 0.$$

Let $H_a(s)$ is the system fn. of an analog filter. This can be expressed in partial fraction form as

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{s - p_k} \quad \text{——(5)}$$

(Laplace Transform).

where, $P_k \rightarrow$ Poles of analog filter

$c_k \rightarrow$ Coefficients in the partial fraction expression.

↳ Inverse Laplace Transform of eqn (5).

$$h_a(t) = \sum_{k=1}^{N} c_k e^{P_k t} \quad t \geq 0 \quad \text{—— (6)}$$

↳ Sample $h_a(t)$ periodically at $t = nT$,

$$h(n) = h_a(nT)$$

$$= \sum_{k=1}^{N} c_k e^{P_k nT} \quad \text{—— (7)}$$

↳ W.K.T $\quad H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \quad \text{—— (8)}$

Sub. eqn (7) in (8)

$$H(z) = \sum_{n=0}^{\infty} \sum_{k=1}^{N} c_k e^{P_k nT} z^{-n}$$

$$= \sum_{k=1}^{N} c_k \sum_{n=0}^{\infty} \left(e^{P_k T} z^{-1}\right)^n$$

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{P_k T} z^{-1}} \quad \text{—— (9).}$$

Imp

If. $H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - P_k}$ then $H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{P_k T} z^{-1}}$

Steps to design a digital filter using Impulse Invariance method

Step 1 :- For the given specifications, find $H_a(s)$ Transfer function of analog filter.

Step 2 :- Select sampling rate of digital filter, T seconds per sample.

Step 3 ⇒ Express analog filter transfer fn.

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{s - p_k}$$

Step 4 ⇒ Compute z- Transform of digital filter.

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{p_k T} z^{-1}}$$

For high sampling rate.

$$H(z) = \sum_{k=1}^{N} \frac{T C_k}{1 - e^{p_k T} z^{-1}} \quad \text{for } T < 1 \text{ second.}$$

Pbm.

Qn For the analog ~~filter~~ transfer function $H(s) = \frac{2}{(s+1)(s+2)}$. Determine $H(z)$ using impulse invariance method. Assume $T = 1$ sec.

Soln. Given $H(s) = \frac{2}{(s+1)(s+2)}$.

Using partial fraction,

$$H(s) = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$A = (s+1)\,H(s) = (s+1)\frac{2}{(s+1)(s+2)}\Big|_{s=-1} \Rightarrow \frac{2}{-1+2} = \frac{2}{1} = 2$$

$$\boxed{A = 2}$$

$B = (S+2)\, H(s)$.

$$= (S+2)\frac{2}{(S+1)(S+2)}\Bigg|_{S=-2} = \frac{2}{S+1}\Bigg|_{S=-2} = \frac{2}{-2+1} = -2.$$

$\boxed{B=-2}$

$$H(s) = \frac{2}{(S+1)} - \frac{2}{(S+2)} \Rightarrow \frac{2}{S-(-1)} - \frac{2}{S-(-2)}$$

Using Impulse Invariant Technique,

$$H(s) = \sum_{k=1}^{N} \frac{C_k}{S-P_k} \Rightarrow H(z) = \sum_{K=1}^{N} \frac{C_K}{1-e^{P_K T} z^{-1}}$$

$$\therefore \frac{2}{S-(-1)} \Rightarrow \frac{2}{1-e^{-1T} z^{-1}}$$

$(S-P_k)$ transformed to $(1-e^{P_K T} z^{-1})$

$$\frac{2}{S-(-2)} \Rightarrow \frac{2}{1-e^{-2T} z^{-1}}$$

$$H(z) = \frac{2}{1-e^{-1} z^{-1}} - \frac{2}{1-e^{-2} z^{-1}}$$

$T = 1$ sec.

$$H(z) = \frac{2}{(1-0.3678 z^{-1})} - \frac{2}{(1-0.1353 z^{-1})}$$

$$H(z) = \frac{0.465 z^{-1}}{1-0.503 z^{-1} + 0.0497 z^{-2}}$$

Prbm
Qn. Apply Impulse invariant method and find $H(z)$ for $H(s) = \frac{s+a}{(s+a)^2+b^2}$

Soln.

$$H(s) = \frac{s+a}{(s+a)^2+b^2} \quad \text{——— (1)}$$

Taking inverse laplace transform.

$$h(t) = \begin{cases} e^{-at}\cos(bt) & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{——— (2)}$$

} Standard

Replace $t = nT$ in eqn (2).

$$h(nT) = \begin{cases} e^{-anT}\cos(bnT) & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$H(z)$?

$$H(z) = \sum_{n=0}^{\infty} h(nT)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} e^{-anT}\cos(bnT)\, z^{-n}$$

$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$.

$$= \sum_{n=0}^{\infty} e^{-anT} z^{-n}\left[\frac{e^{jbnT}+e^{-jbnT}}{2}\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left[e^{-anT}.e^{jbnT} z^{-n} + e^{-anT}e^{-jbnT}z^{-n}\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left[\left[e^{-(a-jb)T}z^{-1}\right]^n + \left[e^{-(a+jb)T}z^{-1}\right]^n\right]$$

In Geometric Progression $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$

$$\therefore = \frac{1}{2}\left[\frac{1}{1-e^{-(a-jb)T}z^{-1}} + \frac{1}{1-e^{-(a+jb)T}z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{\left(1-e^{-(a+jb)T}z^{-1}\right)+\left(1-e^{-(a-jb)T}z^{-1}\right)}{\left(1-e^{-(a-jb)T}z^{-1}\right)\left(1-e^{-(a+jb)T}z^{-1}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{2-z^{-1}\left(e^{-aT}e^{-jbT}+e^{-aT}e^{jbT}\right)}{1-e^{-(a+jb)T}z^{-1}-e^{-(a-jb)T}z^{-1}+\left(e^{-(a-jb)T}z^{-1}\right)\left(e^{-(a+jb)T}z^{-1}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{2-z^{-1}\left(e^{-aT}e^{-jbT}+e^{-aT}e^{jbT}\right)}{1-z^{-1}\left(e^{-(a+jb)T}+e^{-(a-jb)T}\right)+e^{-aT+jbT-aT-jbT}z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{2-z^{-1}e^{-aT}\left(e^{-jbT}+e^{jbT}\right)}{1-z^{-1}e^{-aT}\left(e^{-jbT}+e^{jbT}\right)+e^{-2aT}z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{2-2\cos bT\, z^{-1}e^{-aT}}{1-2\cos bT\, z^{-1}e^{-aT}+e^{-2aT}z^{-2}}\right]$$

$$= \frac{2}{2}\left[\frac{1-\cos bT\, z^{-1}e^{-aT}}{1-2\cos bT\, z^{-1}e^{-aT}+e^{-2aT}z^{-2}}\right]$$

$$H(z) = \frac{1-z^{-1}e^{-aT}\cos bT}{1-2z^{-1}e^{-aT}\cos bT+e^{-2aT}z^{-2}}$$

If $H(s) = \frac{s+a}{(s+a)^2+b^2}$

$$H(z) = \frac{1-\cos bT \cdot z^{-1}e^{-aT}}{1-2\cos bT\, z^{-1}e^{-aT}+e^{-2aT}z^{-2}}$$

If $H(s) = \dfrac{b}{(s+a)^2+b^2}$

Then $H(z) = \dfrac{e^{-aT} \sin bT \, z^{-1}}{1 - 2e^{-aT} \cos bT \, z^{-1} + e^{-2aT} z^{-2}}$

Qn. Convert the analog filter with system transfer fn. $H(s) = \dfrac{s+0.1}{(s+0.1)^2+9}$ into a digital IIR filter by means of impulse invariant method.

Soln. Given, $H(s) = \dfrac{s+0.1}{(s+0.1)^2+9}$

This is in the form of $H(s) = \dfrac{s+a}{(s+a)^2+b^2}$

$$\therefore H(z) = \frac{1 - e^{-aT} \cos bT \, z^{-1}}{1 - 2e^{-aT} \cos bT \, z^{-1} + e^{-2aT} z^{-2}}$$

Here $a = 0.1$ and $b = 3$

Using the above eqn.

$$H(z) = \frac{1 - e^{-0.1T} \cos 3T \, z^{-1}}{1 - 2e^{-0.1T} \cos 3T \, z^{-1} + e^{-2\times0.1T} z^{-2}}$$

Assume $T = 1$ sec

$$H(z) = \frac{1 - e^{-0.1} \cos 3 \, z^{-1}}{1 - 2e^{-0.1} \cos 3 \, z^{-1} + e^{-2\times0.1} z^{-2}}$$

$$H(z) = \frac{1-(-0.8958\,z^{-1})}{1-(-1.7916\,z^{-1}) + 0.8187\,z^{-2}}$$

$$\boxed{H(z) = \frac{1+0.8958\,z^{-1}}{1+1.7916\,z^{-1}+0.8187\,z^{-2}}}$$

Qn. For the analog transfer fn.

$H(s) = \frac{2}{s^2+3s+2}$, determine $H(z)$ using impulse invariant transformation if (a) $T = 1$ second (b) $T = 0.1$ second.

Soln. Given $H(s) = \frac{2}{s^2+3s+2}$

s^2+3s+2 $\quad a=1 \quad b=3 \quad c=2$

$s = -1, -2$

$= \frac{-b \pm \sqrt{b^2-4ac}}{2}$

$= \frac{-3 \pm \sqrt{9-4\times1\times2}}{2}$

$= \frac{-3 \pm \sqrt{9-8}}{2}$

$= \frac{-3+1}{2}$ or $\frac{-3-1}{2}$

$= \frac{-2}{2}$ or $\frac{-4}{2}$

$= -1$ or -2

$$H(s) = \frac{2}{(s+1)(s+2)} \quad \text{——(1)}$$

Partial fraction

$$H(s) = \frac{A}{s+1} + \frac{B}{s+2} \quad \text{——(2)}$$

$$A = (s+1)\,H(s)\Big|_{s=-1}$$

$$= (s+1)\frac{2}{(s+1)(s+2)}\Big|_{s=-1} = \frac{2}{-1+2} = 2$$

$$\boxed{A=2} \quad \text{——(3)}$$

$$B = (s+2)\,H(s)\Big|_{s=-2}$$

$$= (s+2)\frac{2}{(s+1)(s+2)}\Big|_{s=-2} = \frac{2}{-2+1} = -2 \quad \boxed{B=-2} \quad \text{——(4)}$$

Sub. eqn (3) and (4) in (2)

$$H(s) = \frac{2}{s+1} - \frac{2}{s+2}$$

If $H_a(s) = \sum_{k=1}^{N} \frac{C_k}{s - P_k}$ $\quad \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$

$$= \frac{2}{s-(-1)} - \frac{2}{s-(-2)}$$

$$H(z) = \frac{2}{1 - e^{-1 \times T} z^{-1}} - \frac{2}{1 - e^{-2T} z^{-1}}$$

(a) $T = 1$ sec.

$$H(z) = \frac{2}{1 - e^{-1} z^{-1}} - \frac{2}{1 - e^{-2} z^{-1}}$$

$$= \frac{2}{1 - 0.3679 z^{-1}} - \frac{2}{1 - 0.1353 z^{-1}}$$

$$= \frac{2(1 - 0.1353 z^{-1}) - 2(1 - 0.3679 z^{-1})}{(1 - 0.3679 z^{-1})(1 - 0.1353 z^{-1})}$$

$$= \frac{2 - 0.2706 z^{-1} - 2 + 0.7358 z^{-1}}{1 - 0.1353 z^{-1} - 0.3679 z^{-1} + 0.0498 z^{-2}}$$

$$H(z) = \frac{0.4652 z^{-1}}{1 - 0.5032 z^{-1} + 0.0498 z^{-2}}$$

(b) $T = 0.1$ Sec

~~When T≤1 H_N(z) = T×H(z)~~ $T<1 \quad H_N(z) = T \times H(z)$

$$H(z) = \frac{2}{1-e^{-0.1}z^{-1}} - \frac{2}{1-e^{-2\times0.1}z^{-1}} =$$

$$= \frac{2(1-e^{-0.2}z^{-1}) - 2(1-e^{-0.1}z^{-1})}{(1-e^{-0.1}z^{-1})(1-e^{-0.2}z^{-1})}$$

$$= \frac{2 - 1.6375z^{-1} - 2 + 1.8097z^{-1}}{1 - 0.8187z^{-1} - 0.9048z^{-1} + 0.7408z^{-2}}$$

$$H_N(z) = 0.1\left[\frac{0.1722z^{-1}}{1 - 1.7235z^{-1} + 0.7408z^{-2}}\right]$$

$$H_N(z) = \frac{0.01722z^{-1}}{1 - 1.7235z^{-1} + 0.7408z^{-2}}$$

Qn. Using impulse invariance with $T = 1$ sec. Determine $H(z)$ if $H(s) = \frac{1}{s^2+\sqrt{2}s+1}$

In impulse invariance denominator ⇒ perfect square.

Given $H(s) = \frac{1}{s^2+\sqrt{2}s+1}$

To make perfect square. ⇒ Two factors.
Middle term ⇒ coefficient with s.

$$\left(\frac{\text{Coefficient}}{2}\right)^2$$

$$H(s) = \frac{1}{s^2+\sqrt{2}s+1+\left(\frac{\sqrt{2}}{2}\right)^2-\left(\frac{\sqrt{2}}{2}\right)^2}$$

$$= \frac{1}{s^2+\sqrt{2}s+1+\left(\frac{\sqrt{2}}{\sqrt{2}\times\sqrt{2}}\right)^2-\left(\frac{\sqrt{2}}{\sqrt{2}\times\sqrt{2}}\right)^2}$$

$$= \frac{1}{s^2+\sqrt{2}s+1+\left(\frac{1}{\sqrt{2}}\right)^2-\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \frac{1}{s^2+\sqrt{2}s+\left(\frac{1}{\sqrt{2}}\right)^2+1-\frac{1}{2}}$$

$(a+b)^2 = a^2+2ab+b^2$

$$H(s) = \frac{1}{\left(s+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}$$

$$H(s) = \frac{1}{\left(s+\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}$$

If $H(s) = \frac{b}{(s+a)^2+b^2}$

Then $H(z) = \frac{e^{-aT}\sin bT\, z^{-1}}{1-2e^{-aT}\cos bT\, z^{-1}+e^{-2aT}z^{-2}}$

$$H(s) = \frac{\sqrt{2}\times\frac{1}{\sqrt{2}}}{\left(s+\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$H(s) = \sqrt{2}\left[\frac{\frac{1}{\sqrt{2}}}{\left(s+\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\right]$$

$$H(z) = \frac{\sqrt{2}\left[e^{-\frac{1}{\sqrt{2}}T} \sin \frac{1}{\sqrt{2}}T \, z^{-1}\right]}{1 - 2e^{-\frac{1}{\sqrt{2}}T} \cos\left(\frac{1}{\sqrt{2}}T\right) z^{-1} + e^{-2\frac{1}{\sqrt{2}}T} z^{-2}}$$

T=1 Sec

$$H(z) = \frac{\sqrt{2}\left[e^{-\frac{1}{\sqrt{2}}} \sin\left(\frac{1}{\sqrt{2}}\right) z^{-1}\right]}{1 - 2e^{-1/\sqrt{2}} \cos \frac{1}{\sqrt{2}} z^{-1} + e^{-\sqrt{2}} z^{-2}}$$

$$= \frac{\sqrt{2}\left(0.4931 \times 0.6496 \, z^{-1}\right)}{1 - 0.9861 \times 0.7602 \, z^{-1} + 0.2431 z^{-2}}$$

$$\boxed{H(z) = \frac{0.4530 z^{-1}}{1 - 0.7496 z^{-1} + 0.2431 z^{-2}}}$$

~~B1~~ IIR FILTER DESIGN BY BILINEAR TRANSFORMATION

- Invariance technique ⇒ Severe limitation
  ⇓
  Appropriate only for band limited signal.

- In Impulse invariance mapping → a strip in s-plane b/w $\Omega = (2k-1)\frac{\pi}{T}$ and $\Omega = (2k+1)\frac{\pi}{T}$ is mapped completely into the z-plane.

- Therefore, the aliasing is caused by the mapping of all such strips on tip of each other in z-plane.

- To overcome aliasing problem → need a transformation that is one-to-one mapping.
  ⇒ i.e., maps a single point in the s-plane to a unique point in the z-plane.

- This type of mapping is called Linear transformation

- Bilinear transformation is a conformal mapping that transforms the $j\Omega$ axis into the unit circle in the z-plane only once → this avoids the aliasing effect.

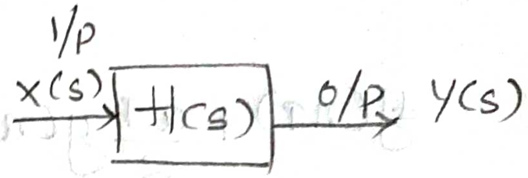
- The points in the left half of the s-plane → mapped inside the unit circle in z-plane

- Points in the right half of the s-plane → mapped ~~into the~~ outside the unit-circle in z-plane.

↳ Consider an analog filter with system fn.

$$H(s) = \frac{b}{s+a} \quad \text{——(1)}$$

↳ Written as



$$\frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

$$H(s) = \frac{O/P}{I/P} = \frac{Y(s)}{X(s)}$$

$$Y(s)(s+a) = X(s)\, b.$$

$$s\,Y(s) + a\,Y(s) = b\,X(s) \quad \text{——(2)}$$

↳ This can be characterized by the differential eqn.

$$\frac{dy(t)}{dt} + a\,y(t) = b\,x(t). \quad \text{——(3)}$$

↳ y(t) can be approximated by the trapezoidal formula.

$$y(t) = \int_{t_0}^{t} y'(t)\,dt + y(t_0) \quad \text{——(4)}.$$ General soln.

where $y'(t) \rightarrow$ derivative of $y(t)$.

↳ $t = nT$ and $t_0 = nT - T$ in eqn (4).

$$y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT-T)\right] + y(nT-T). \quad \text{——(5)}$$

↳ Sub. $t = nT$ in eqn (3).
$t = (nT-T)$

$$\left.\begin{aligned} y'(nT) &= -a\,y(nT) + b\,x(nT) \\ y'(nT-T) &= -a\,y(nT-T) + b\,x(nT-T) \end{aligned}\right\} \text{——(6)}$$

↳ Sub. eqn (6) in (5)

$$y(nT) = \frac{T}{2}\left[-a\,y(nT) + b\,x(nT) - a\,y(nT-T) + b\,x(nT-T)\right] + y(nT-T)$$

$$y(nT) + \frac{aT}{2} y(nT) + \frac{aT}{2} y(nT-T) - y(nT-T)$$
$$= \frac{T}{2} bx(nT) + \frac{T}{2} bx(nT-T)$$

$$y(nT) + \frac{aT}{2} y(nT) - \left(1 - \frac{aT}{2}\right) y(nT-T)$$
$$= \frac{bT}{2}\left(x(nT) + x(nT-T)\right) \quad \text{——— (7)}$$

↳ sub. $y(nT) = y(n)$ & $x(nT) = x(n)$

$y(nT-T) = y(n-1)$ & $x(nT-T) = x(n-1)$

↳ eqn (7) becomes

$$\left(1+\frac{aT}{2}\right)y(n) - \left(1-\frac{aT}{2}\right)y(n-1) = \frac{bT}{2}\left[x(n) + x(n-1)\right] \quad \text{(8)}$$

↳ Z-transform of eqn (8).

$$\left(1+\frac{aT}{2}\right)Y(z) - \left(1-\frac{aT}{2}\right)z^{-1}Y(z) = \frac{bT}{2}\left[1+z^{-1}\right]X(z)$$

$$Y(z)\left[\left(1+\frac{aT}{2}\right) - \left(1-\frac{aT}{2}\right)z^{-1}\right] = \frac{bT}{2}\left[1+z^{-1}\right]X(z).$$

↳ The system function of the digital filter.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{bT}{2}\left(1+z^{-1}\right)}{\left(1+\frac{aT}{2}\right) - \left(1-\frac{aT}{2}\right)z^{-1}}$$

$$H(z) = \frac{\frac{bT}{2}(1+z^{-1})}{1+\frac{aT}{2} - z^{-1} + \frac{aT}{2}z^{-1}}$$

$$H(z) = \frac{\frac{bT}{2}(1+z^{-1})}{(1-z^{-1}) + \frac{aT}{2}(1+z^{-1})}$$

↳ dividing numerator and ~~nume~~ denominator of above eqn by $\frac{T}{2}(1+z^{-1})$

$$H(z) = \frac{\frac{bT}{2}(1+z^{-1})}{(1-z^{-1}) + \frac{aT}{2}(1+z^{-1})} \times \frac{\frac{T}{2}(1+z^{-1})}{\frac{T}{2}(1+z^{-1})}$$

Solving,

H(s)= b/(s+a)

$$H(z) = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a} \quad \text{——— (9)}$$

↳ Comparing eqn (1) and (9)

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \quad \text{——— (10)}$$

↳ This relationship between s and z is known as Bilinear transformation.

Let $z = re^{j\omega}$ and $s = \sigma + j\Omega$. ——— (11)

↳ eqn (10) can be expressed as

$$s = \frac{2(z-1)}{T(z+1)}$$

$$= \frac{2}{T}\left[\frac{re^{j\omega}-1}{re^{j\omega}+1}\right] = \frac{2}{T}\left[\frac{r\cos\omega + jr\sin\omega - 1}{r\cos\omega + jr\sin\omega + 1}\right]$$

Re-arrange,

$$s = \frac{2}{T}\left[\frac{r\cos\omega - 1 + jr\sin\omega}{r\cos\omega + 1 + jr\sin\omega}\right]$$

Multiplying Deno. & Num. by $(r\cos\omega + 1 - jr\sin\omega)$

$$s = \frac{2}{T}\left[\frac{r\cos\omega - 1 + jr\sin\omega}{r\cos\omega + 1 + jr\sin\omega}\right] \times \left[\frac{r\cos\omega + 1 - jr\sin\omega}{r\cos\omega + 1 - jr\sin\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r^2\cos^2\omega - 1 + r^2\sin^2\omega + j2r\sin\omega}{(r\cos\omega + 1)^2 + r^2\sin^2\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r^2\cos^2\omega - 1 + r^2\sin^2\omega + j2r\sin\omega}{1 + r^2\cos^2\omega + 2r\cos\omega + r^2\sin^2\omega}\right]$$

↳ Separating imaginary and real part

$$s = \frac{2}{T}\left[\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega} + j\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right] \quad \text{——— (12)}$$

↳ Comparing eqn (11) and (12)

$$\sigma = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega}\right] \quad \text{and} \quad \Omega = \frac{2}{T}\left[\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right] \quad \text{— (13)}$$

↳ if $r \leq 1$ then $\sigma < 0$

$r > 1$, then $\sigma > 0$

$r = 1$, then $\sigma = 0$.

$$\therefore \Omega = \frac{2}{T}\frac{\sin\omega}{1+\cos\omega} = \frac{2}{T}\left(\frac{2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{2\cos^2\frac{\omega}{2}}\right)$$

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2} \quad \text{— (14)}$$

$$\omega = 2\tan^{-1}\frac{\Omega T}{2} \quad \text{— (15)}$$

Comparison between FIR and IIR filters.

	FIR filter	IIR filter
1.	The impulse response of this filter is restricted to finite number of samples.	The impulse response of this filter extends over an infinite duration.
2.	FIR filters can have precisely linear phase.	These filters do not have linear phase.
3.	Closed-form design equations do not exist.	A variety of frequency selective filters can be designed using closed-form design formulas.
4.	Most of the design methods are iterative procedures, requiring powerful computational facilities for their implementation.	These filter can be designed using only a hand calculator and tables of analog filter design parameters.
5.	Greater flexibility to control the shape of their magnitude response.	Less flexibility specially for obtaining non-standard frequency responses.
6.	In these filters, the poles are fixed at the origin, high selectivity can be achieved by using a relatively high order for the transfer function.	The poles are placed anywhere inside the unit circle, high selectivity can be achieved with low-order transfer functions.
7.	Always stable.	Not always stable.
8.	Errors due to roundoff noise are less severe.	IIR filters are more susceptible to errors due to roundoff noise.

MODULE – IV

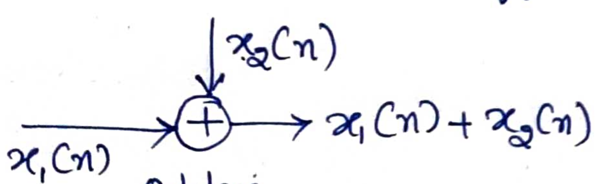
REALIZATION OF FIR FILTER (Discrete Time System).

Block diagram representation of linear constant coefficient difference eqn.

- For designing of digital filters → the system function H(z) or the corresponding impulse response h(n) should be specified.
- After that we can implement the digital filter structure in hardware or software form with the help of its difference eqn. → which is directly obtained from the system fn. H(z) or impulse response h(n).
- Each difference eqn. or Computational algorithm may be implemented simply by using a digital Computer or a particular digital hardware or a particular programmable integrated Circuit.
- The basic elements required for implementation of an LTI discrete time system are adder, multipliers and memory for storing delayed sequence values and coefficients.

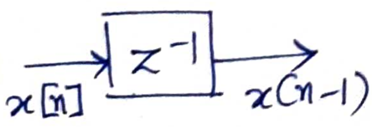
arrow mark → imp

(i)



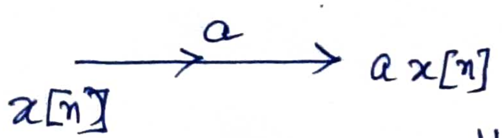
Adder

(iii)



Unit delay

(ii)



multiplier

↳ In digital implementation, the delay operation can be implemented by providing a storage register by each unit delay if required. So sometimes the delay operations is termed as delay register.

FIR FILTER REALIZATION

↳ FIR filters can be realised through various methods such as Direct form, Cascade form, Parallel form and Lattice structure.

(i) Direct form realization.

The direct form of FIR may be obtained by using the eqn of linear convolution

$$y[k] = \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k).$$

If we consider there are N samples then the equation becomes

$$y[n] = \sum_{k=0}^{N-1} h(k)\, x(n-k)$$

(h(k) → coefficient constant; x(n−k) → i/p)

$$y[n] = h(0)\,x(n) + h(1)\,x(n-1) + h(2)\,x(n-2) + \dots$$
$$\dots + h(N-1)\,x(n-N-1).$$

z-Transform

$$Y[z] = h(0)\,X(z) + h(1)\,X(z)\,z^{-1} + h(2)\,X(z)\,z^{-2} +$$
$$\dots + h(N-1)\,X(z)\,z^{-(N-1)}.$$

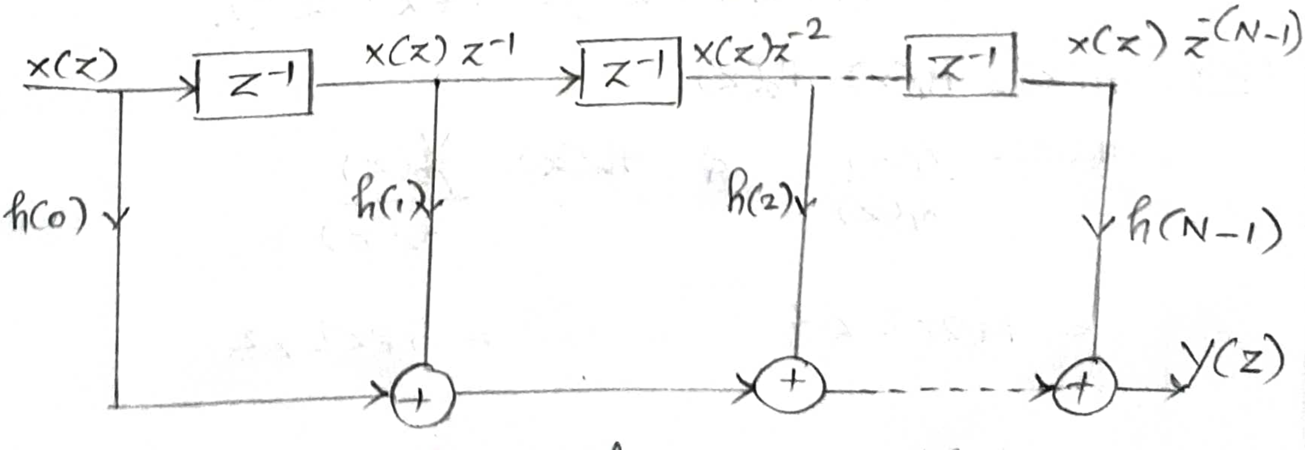


Fig - Direct form Realisation

Pbm

Qn. Determine the direct form realisation of System fn.
$H(z) = 1+2z^{-1} - 3z^{-2} - 4z^{-3} + 5z^{-4}$.

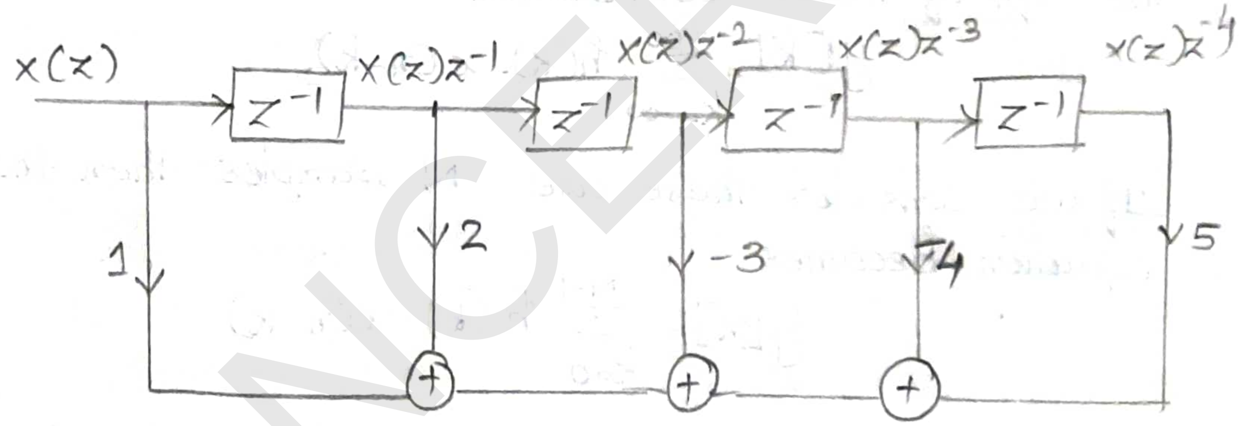
Soln. Given

$H(z) = 1+2z^{-1} - 3z^{-2} - 4z^{-3} + 5z^{-4}$

$H(z) = \frac{Y(z)}{X(z)}$.

$Y(z) = [1+2z^{-1} - 3z^{-2} - 4z^{-3} + 5z^{-4}]\, X(z)$

$Y(z) = X(z) + 2X(z)z^{-1} - 3X(z)z^{-2} - 4X(z)z^{-3} + 5X(z)z^{-4}$



(ii) Cascade Form Realisation of FIR Filter.
fn → Combine

Qn. Obtain the cascade realisation of System fn.
$H(z) = (1+2z^{-1} - z^{-2})(1+z^{-1} - z^{-2})$.

Soln. $H(z) = H_1(z)\, H_2(z)$

$H_1(z) = \frac{Y_1(z)}{X_1(z)}$ & $H_2(z) = \frac{Y_2(z)}{X_2(z)}$

$= 1+2z^{-1} - z^{-2}$ $\qquad = 1+z^{-1} - z^{-2}$

$\therefore \; Y_1(z) = (1 + 2z^{-1} - z^{-2}) X_1(z)$

$\quad = \underset{X(z)}{X_1(z)} + 2z^{-1}X_1(z) - z^{-2}X_1(z).$

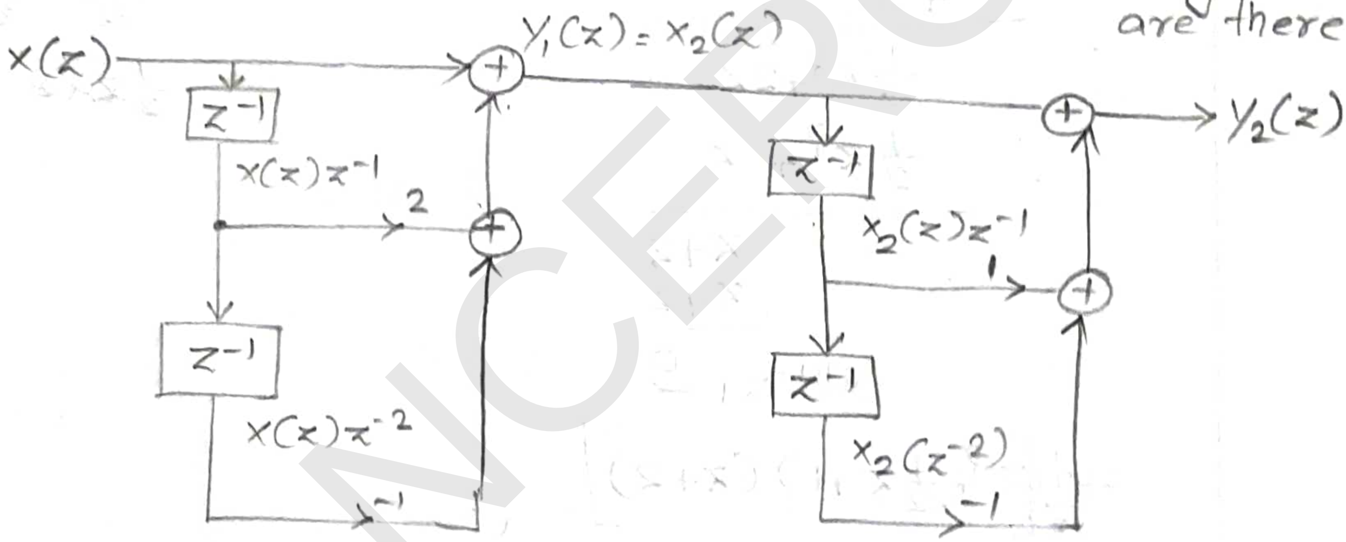
$Y_2(z) = (1 + z^{-1} - z^{-2}) X_2(z)$

$\quad = X_2(z) + X_2(z)z^{-1} - X_2(z)z^{-2}$

(Note: 2 fn in Qn → 1st fn o/p ⇒ in/p of 2nd fn.)

$X_1(z) = X(z)$ $\qquad$ $\underset{\text{o/p of 1st fn.}}{Y_1(z)} = \underset{\text{2nd fn. in/p}}{X_2(z)}$

$Y_1(z)$ = Count how many z^{-1} are there



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Qn. Obtain the cascade realisation of system fn.

$H(z) = 1 + \frac{5}{2}z^{-1} + 2z^{-2} + 2z^{-3}$

Given, $1 + \frac{5}{2}z^{-1} + 2z^{-2} + 2z^{-3}$ $\qquad$ High Power $\frac{z^3}{z^3}$

$H(z) = \frac{z^3}{z^3}\left[1 + \frac{5}{2}z^{-1} + 2z^{-2} + 2z^{-3}\right]$

$= \frac{1}{z^3}\left[z^3 + \frac{5}{2}z^2 + 2z^1 + 2\right]$ $\qquad$ [By trial & Error]

(By Trial and error)

$H(z) = \frac{1}{z^3}\left[z^3 + \frac{5}{2}z^2 + 2z^1 + 2\right]$

$z=1 \Rightarrow 1^3 + \frac{5}{2} + 2 + 2 \neq 0$

$z=-2 \Rightarrow (-2)^3 + \frac{5}{2}\times(-2)^2 + 2\times -2 + 2$

$-8 + 10 - 4 + 2 = 0$

$\Rightarrow \boxed{z+2}$

$$\begin{array}{r} z^2 + \frac{1}{2}z + 1 \\ z+2 \,\big)\, z^3 + \frac{5}{2}z^2 + 2z^1 + 2 \\ z^3 + 2z^2 \\ \hline \frac{1}{2}z^2 + 2z \\ \frac{1}{2}z^2 + z \\ \hline z+2 \\ z+2 \\ \hline 0 \end{array}$$

$\frac{5}{2} - 2 = \frac{5-4}{2} = \frac{1}{2}$

$\frac{1}{2}z \times z$

$\Rightarrow \frac{1}{z^3}\left[(z^2 + \frac{1}{2}z + 1)(z+2)\right]$.

$H(z) = \frac{1}{z^3}\left[(z+2)(z^2 + \frac{1}{2}z + 1)\right]$

↳ To make it in $z^{-1} \rightarrow \frac{1}{z^3}$ multiply inside.

$H(z) = \frac{1}{z^3}\left[z(1 + \frac{2}{z})\ z^2(1 + \frac{1}{2z} + \frac{1}{z^2})\right]$

$= \frac{z^3}{z^3}\left[(1 + 2z^{-1})(1 + \frac{1}{2}z^{-1} + z^{-2})\right]$

$H(z) = (1 + 2z^{-1})(1 + \frac{1}{2}z^{-1} + z^{-2})$

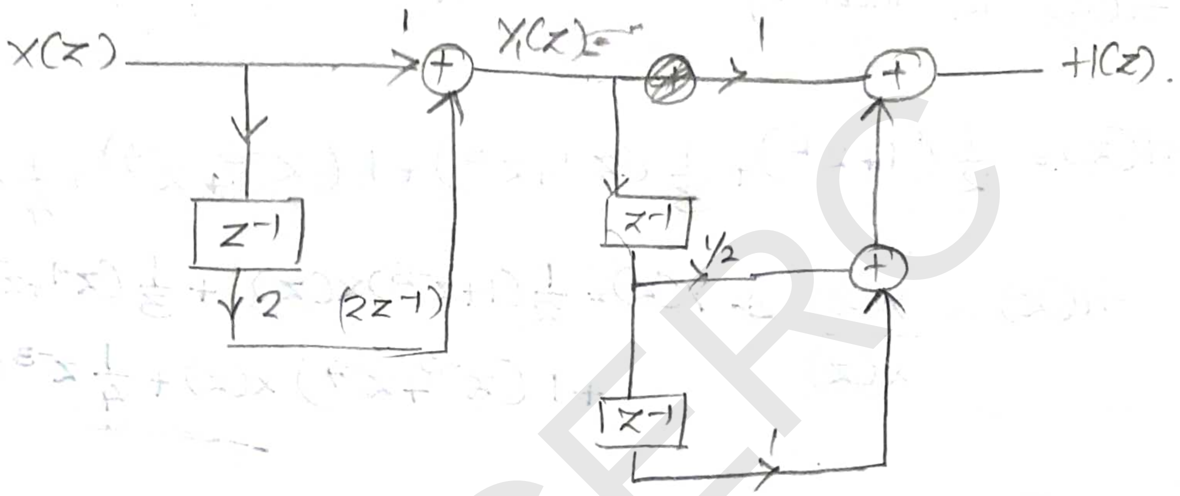
$H_1(z) \qquad H_2(z)$

$$H_1(z) = \frac{Y_1(z)}{X_1(z)} = 1+2z^{-1}$$

$$H_2(z) = \frac{Y_2(z)}{X_2(z)} = 1+\frac{1}{2}z^{-1}+z^{-2}$$

$$Y_1(z) = X_1(z)\left[1+2z^{-1}\right] = X_1(z)+2z^{-1}X_1(z) \qquad X_1(z) = X(z)$$

$$Y_2(z) = X_2(z)\left[1+\frac{1}{2}z^{-1}+z^{-2}\right] = X_2(z)+\frac{1}{2}z^{-1}X_2(z)+z^{-2}X_2(z)$$



cascade form.

(iii) Linear Phase Realisation

Qn. Realize the system fn.

$$H(z) = \frac{1}{2}+\frac{1}{3}z^{-1}+z^{-2}+\frac{1}{4}z^{-3}+z^{-4}+\frac{1}{3}z^{-5}+\frac{1}{2}z^{-6}$$

Soln. check linear or not ?

Given $H(z) = \underset{h(0)}{\frac{1}{2}}+\underset{h(1)}{\frac{1}{3}}z^{-1}+\underset{h(2)}{1}\cdot z^{-2}+\underset{h(3)}{\frac{1}{4}}z^{-3}+\underset{h(4)}{1}\cdot z^{-4}+\underset{h(5)}{\frac{1}{3}}z^{-5}+\underset{h(6)}{\frac{1}{2}}z^{-6}$

Linear phase filter Cond^n $\Rightarrow$ $h(n) = h(N-1-n)$.

7 terms $\therefore$ $N=7$

$$h(0) = h(7-1-0) = h(6)$$

$$h(1) = h(7-1-1) = h(5)$$

$$h(2) = h(4)$$

$$h(3)$$

$h(0) = h(6) = \frac{1}{2}$

$h(1) = h(5) = \frac{1}{3}$

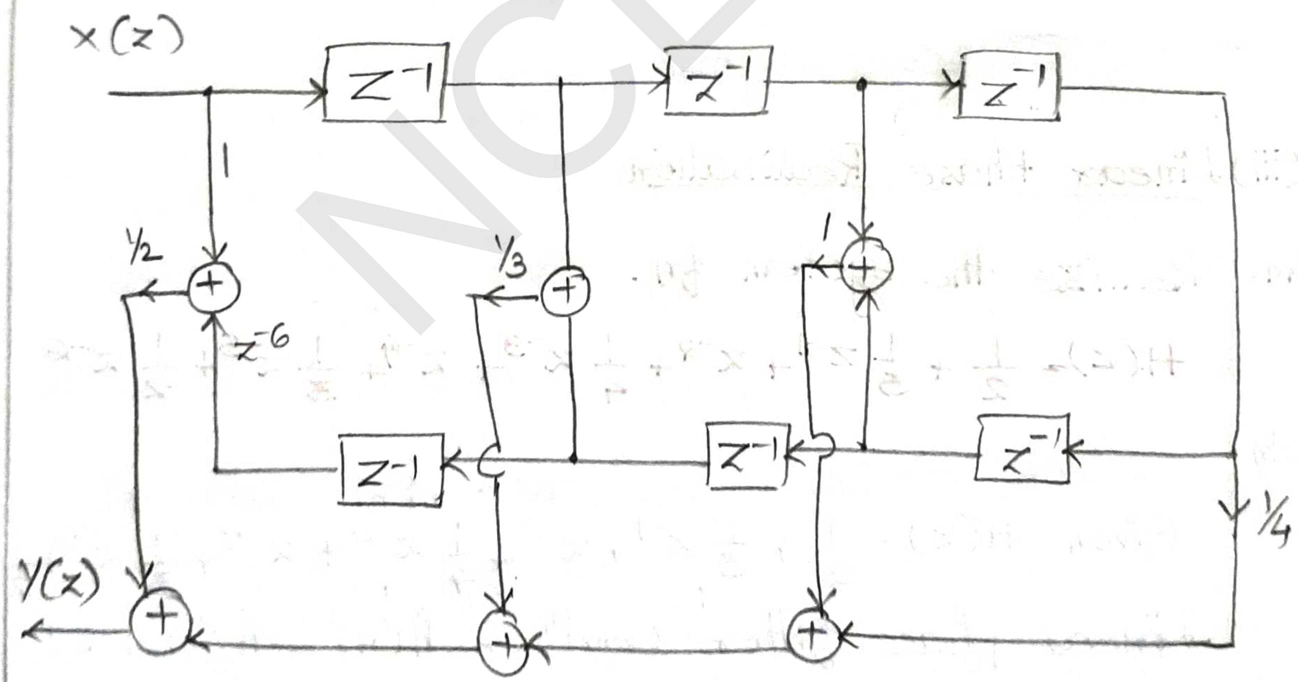
$h(2) = h(4) = 1$

$h(3) = \frac{1}{4}$

By inspection we find that the system fn. $H(z)$ is that of a linear phase FIR Filter.

$$H(z) = \frac{1}{2}(1+z^{-6}) + \frac{1}{3}(z^{-1}+z^{-5}) + 1(z^{-2}+z^{-4}) + \frac{1}{4}z^{-3}$$

$$H(z) = \frac{Y(z)}{X(z)} \Rightarrow Y(z) = \frac{1}{2}(1+z^{-6})X(z) + \frac{1}{3}(z^{-1}+z^{-5})X(z) + 1(z^{-2}+z^{-4})X(z) + \frac{1}{4}z^{-3}X(z).$$



REALIZATION OF IIR FILTERS.

$y(n) \Rightarrow y(n-1), x(n), x(n-1)$

↳ IIR Filter ⇒ Recursive realization

↳ Recursive realization ⇒ Current o/p sample $y(n)$
→ fn. of past o/ps, past and present in/ps.
Poles and zeros.

↳ FIR filter ⇒ Non-recursive realization
Current o/p sample $y(n)$ ⇒ fn of only past and present in/ps. $y(n) \Rightarrow x(n-1)$ & $x(n)$.
↳ (zeros)

↳ IIR filter can be realized in many forms.
(1) Direct form - I realization
(2) Direct form - II realization
(3) Transposed direct form realization
(4) Cascade form realization
(5) Parallel form realization.

(1.) Direct Form - I Realization

Consider an LTI recursive system described by the difference eqn.

$$y(n) = -\sum_{k=1}^{N} a_k \underbrace{y(n-k)}_{\text{Past o/p}} + \sum_{k=0}^{M} b_k \underbrace{x(n-k)}_{\text{Past in/p}} \quad —(1)$$

$$\Rightarrow -a_1 y(n-1) - a_2 y(n-2) - \ldots - a_N y(n-N)$$

$$\Rightarrow b_0 x(n) + b_1 x(n-1) + \ldots b_M x(n-M) = \omega_n$$

Eqn (1) becomes

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - \ldots - a_N y(n-N) + \omega_n.$$

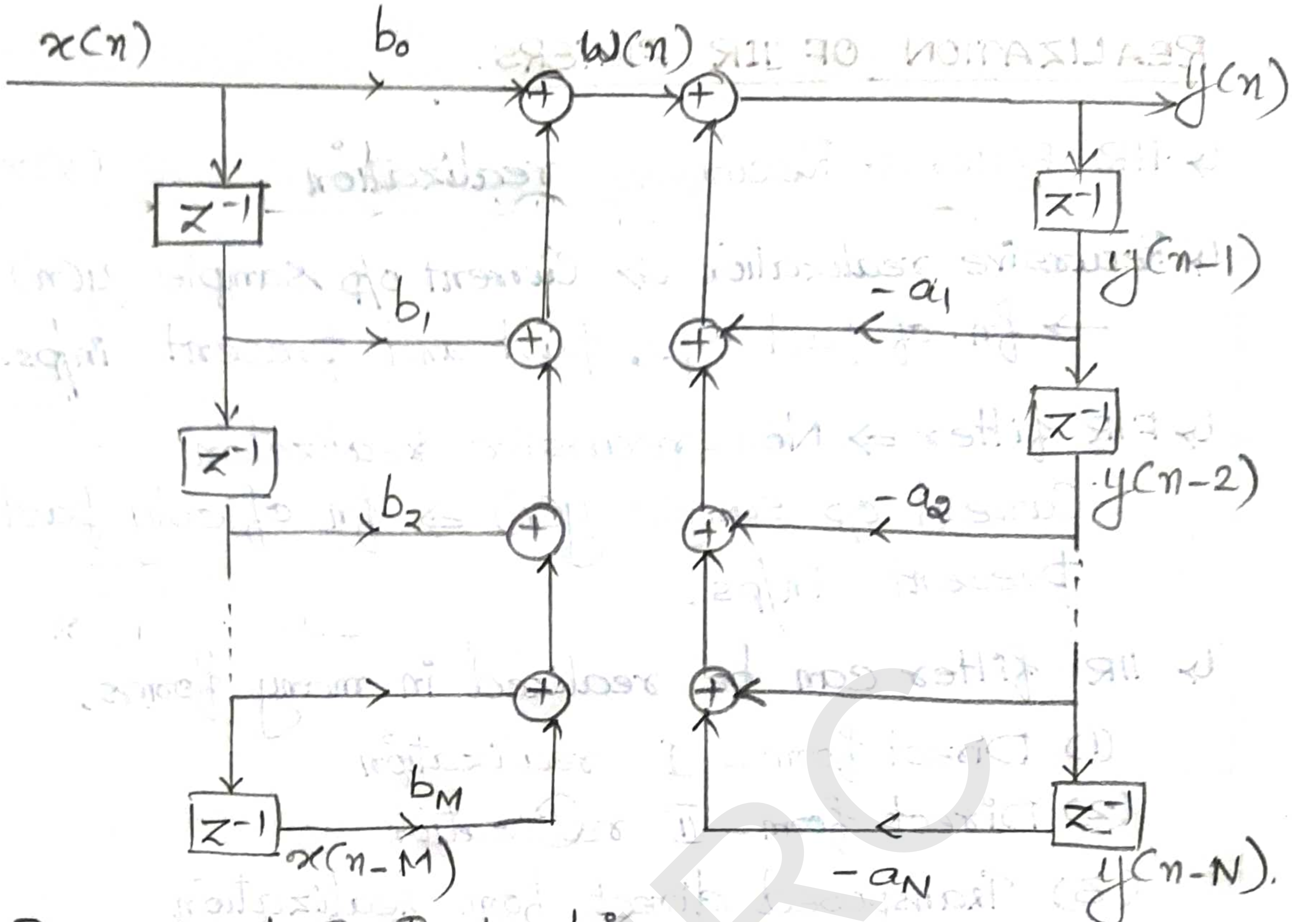


Fig :- Direct - I Realization

Pblm

Qn. Realize the second order digital filter

$$y(n) = 2r\cos(\omega_0)\,y(n-1) - r^2 y(n-2) + x(n) - r\cos(\omega_0)\,x(n-1).$$

Using Direct form - I realization

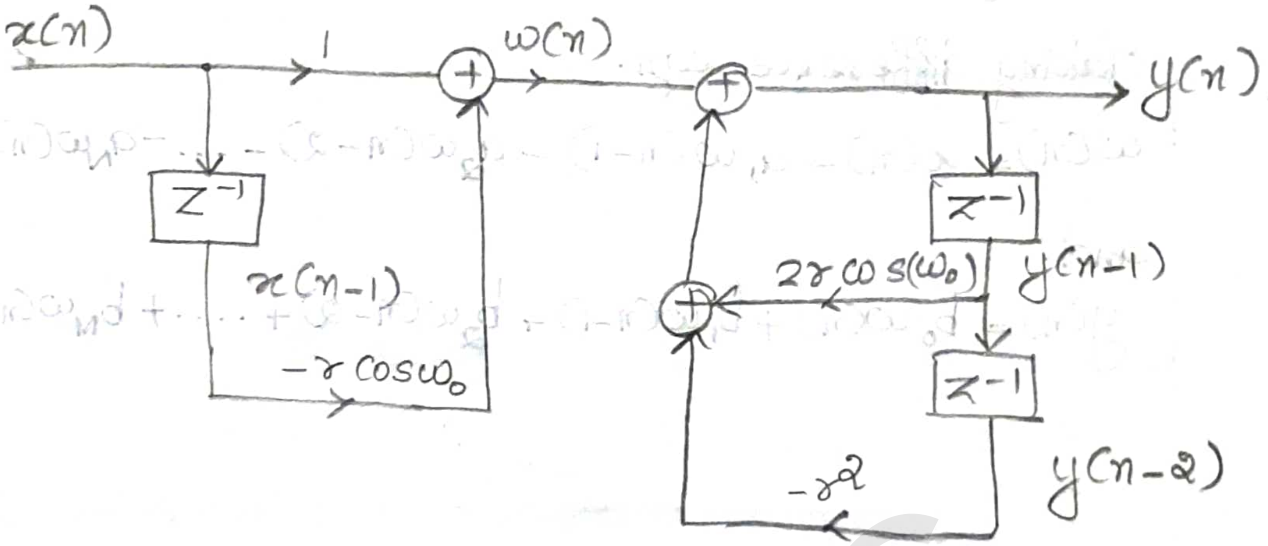
Soln.

$$y(n) = \underbrace{2r\cos(\omega_0)\,y(n-1) - r^2 y(n-2)}_{\text{past o/p}} + \underbrace{x(n) - r\cos(\omega_0)\,x(n-1)}_{\text{i/p}}.$$

Let $x(n) + r\cos(\omega_0)\,x(n-1) = \omega(n)$

then $y(n) = 2r\cos(\omega_0)\,y(n-1) - r^2 y(n-2) + \omega(n).$

~~Realize w(n).~~



(2) Direct form - II realization

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k).$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}.$$ standard

Let $\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$

$$\frac{Y(z)}{W(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

~~W(z) = X(z) [1 −~~ $W(z)\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right) = X(z)$

Solving.

$$W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) \ldots - a_N z^{-N} W(z)$$

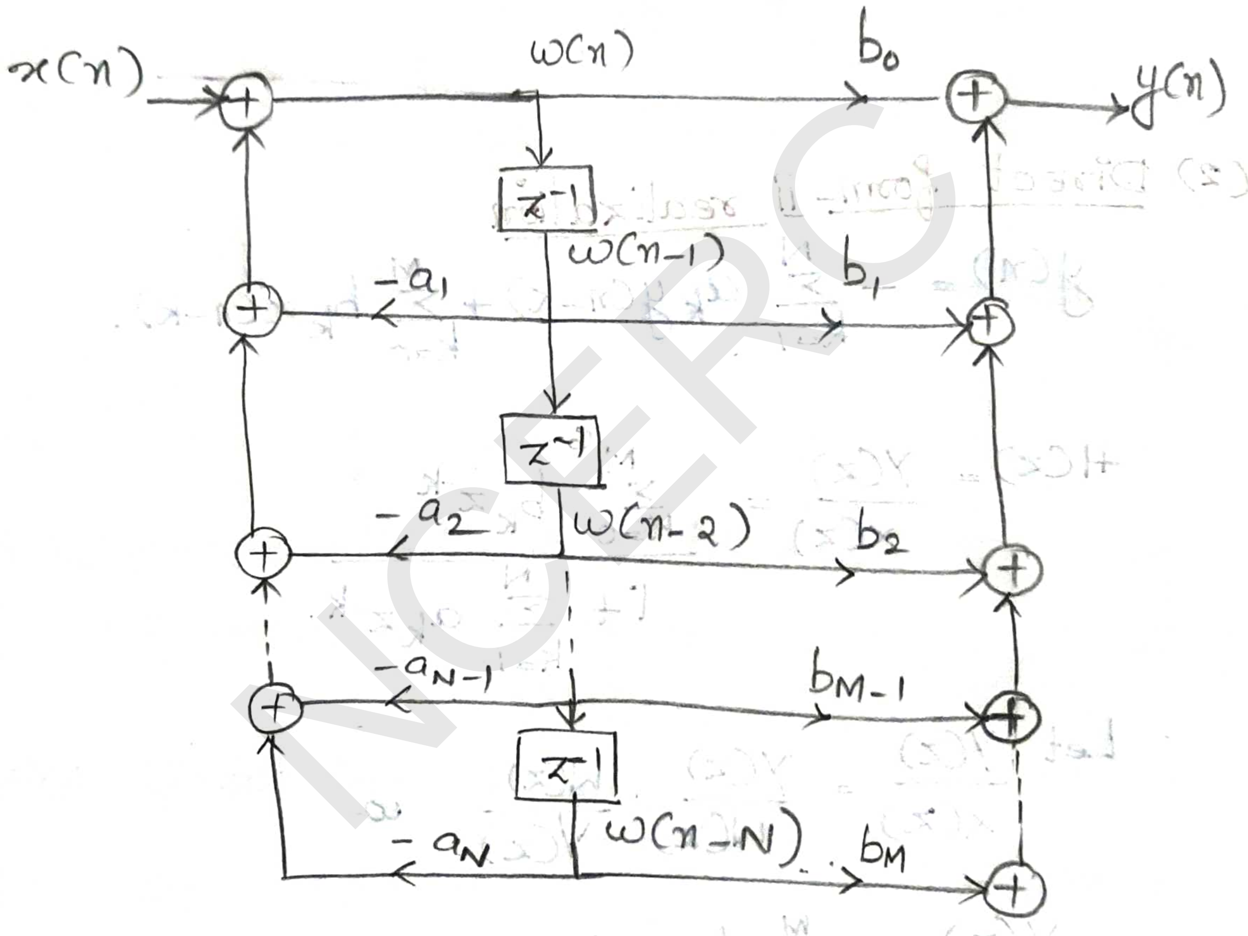
Taking $Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + \ldots b_M z^{-M} W(z)$.

Taking difference eqn.

$$\omega(n) = x(n) - a_1 \omega(n-1) - a_2 \omega(n-2) - \ldots - a_N \omega(n-N)$$

and.

$$y(n) = b_0 \omega(n) + b_1 \omega(n-1) + b_2 \omega(n-2) + \ldots + b_M \omega(n-M).$$



pbm

Qn. Determine the direct form II realization for the following system

$$y(n) = -0.1\,y(n-1) + 0.72\,y(n-2) + 0.7\,x(n) - 0.252\,x(n-1)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}.$$

$$y(n) = -0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.252x(n-1)$$

Taking z Transform.

$$Y(z) = -0.1z^{-1}Y(z) + 0.72z^{-2}Y(z) + 0.7X(z) - 0.252z^{-1}X(z).$$

$$Y(z) + 0.1z^{-1}Y(z) - 0.72z^{-2}Y(z) = 0.7X(z) - 0.252z^{-1}X(z)$$

$$Y(z)\left[1 + 0.1z^{-1} - 0.72z^{-2}\right] = X(z)\left[0.7 - 0.252z^{-1}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{0.7 - 0.252z^{-1}}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

$$\Rightarrow \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

$$\frac{Y(z)}{W(z)} = 0.7 - 0.252z^{-1} \Rightarrow Y(z) = W(z)\left[0.7 - 0.252z^{-1}\right]$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 + 0.1z^{-1} - 0.72z^{-2}} \Rightarrow X(z) = W(z)\left[1 + 0.1z^{-1} - 0.72z^{-2}\right]$$

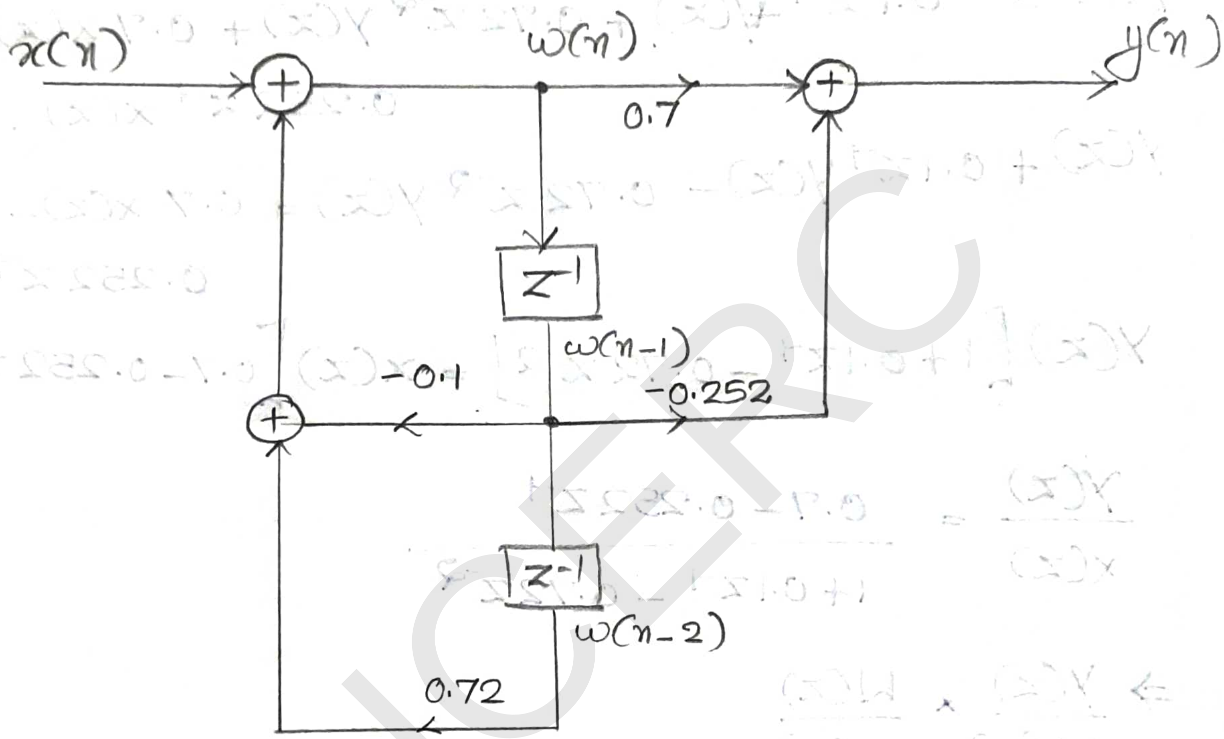
$$X(z) = W(z) + 0.1z^{-1}W(z) - 0.72z^{-2}W(z)$$

$$W(z) = X(z) - 0.1z^{-1}W(z) + 0.72z^{-2}W(z)$$

Taking difference eqn.

$$w(n) = x(n) - 0.1\, w(n-1) + 0.72\, w(n-2) \quad \text{——(1)}$$

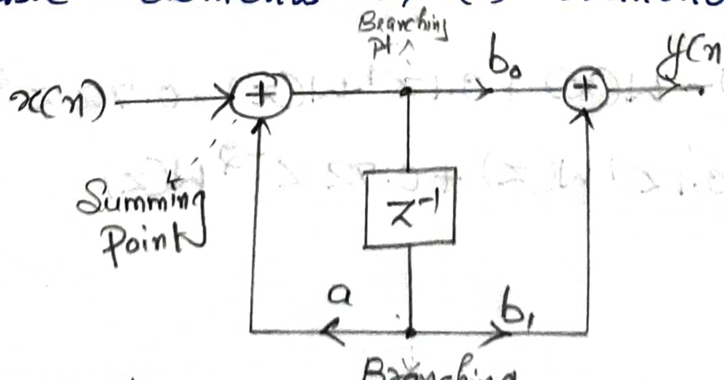
$$y(n) = w(n)\, 0.7 - 0.252\, w(n-1).$$



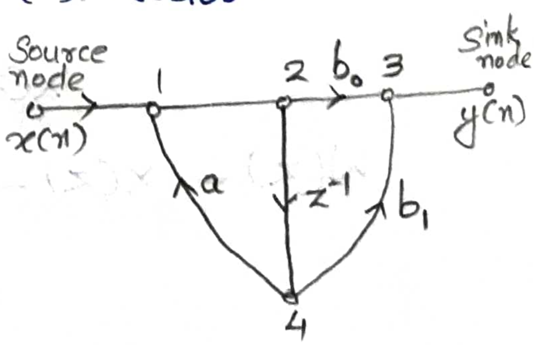
Signal Flowgraph

Signal flow Graph ⇒ Graphical representation of the relationship b/w the variables of a set of linear differences equations.

Basic elements ⇒ (1) Branches (2) Nodes.



Block diagram



Signal Flow Graph

TRANSPOSITION THEOREM AND TRANSPOSED STRUCTURE

Transpose of a structure is defined by the following operations.

(i) Reverse the direction of all branches in the signal flow graph

(ii) Interchange the inputs and o/ps.

(iii) Reverse the roles of all nodes in the flowgraph

(iv) Summing points become branching points

(v) Branching points become summing points.

According to transposition theorem, the system transfer fn. remain unchanged by transposition.

pbm

Qn. Determine the direct form II and Transposed direct-II form-II for the gn. system.

$$y(n) = \frac{1}{2}y(n-1) - \frac{1}{4}y(n-2) + x(n) + x(n-1)$$

Soln.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

$$Y(z) = \frac{1}{2}z^{-1}Y(z) - \frac{1}{4}z^{-2}Y(z) + X(z) + z^{-1}X(z)$$

$$Y(z) - \frac{1}{2}z^{-1}Y(z) + \frac{1}{4}z^{-2}Y(z) = X(z) + z^{-1}X(z)$$

$$Y(z)\left[1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}\right] = X(z)\left[1 + z^{-1}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}}$$

$$\frac{Y(z)}{W(z)} \times \frac{W(z)}{X(z)}$$

$$\frac{Y(z)}{W(z)} = 1+z^{-1} \quad \text{and} \quad \frac{W(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}$$

$$Y(z) = W(z)(1+z^{-1})$$

$$Y(z) = W(z) + z^{-1}W(z) \quad \text{—— (1)}$$

$$W(z)\left[1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}\right) = X(z)$$

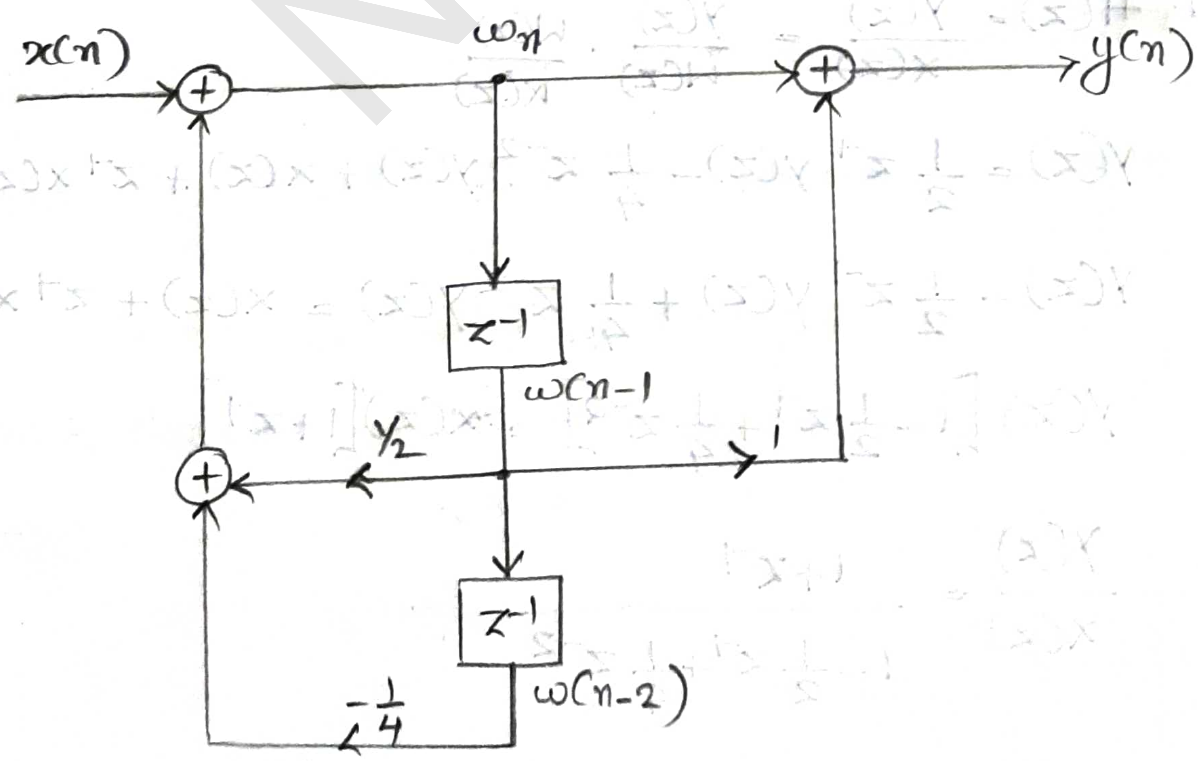
$$W(z) - \frac{1}{2}z^{-1}W(z) + \frac{1}{4}z^{-2}W(z) = X(z)$$

$$W(z) = X(z) + \frac{1}{2}z^{-1}W(z) - \frac{1}{4}z^{-2}W(z). \quad \text{—— (2)}$$

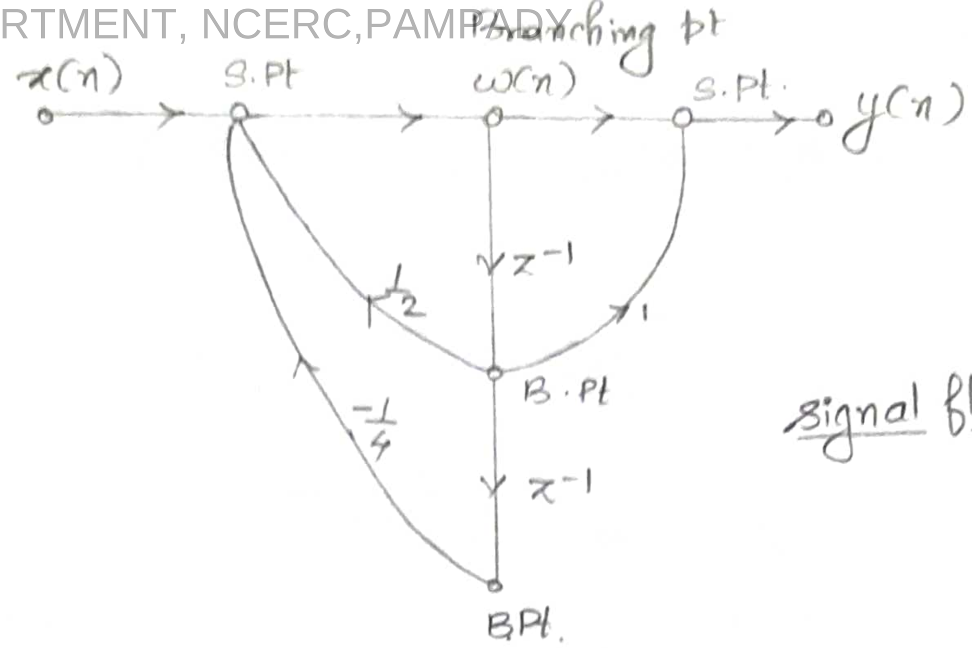
Taking difference eqn. on eqn (2) & (1).

$$w(n) = x(n) + \frac{1}{2}w(n-1) - \frac{1}{4}w(n-2) \quad \text{—— (3)}$$

$$y(n) = w(n) + w(n-1) \quad \text{—— (4)}$$



Direct form-II realization.



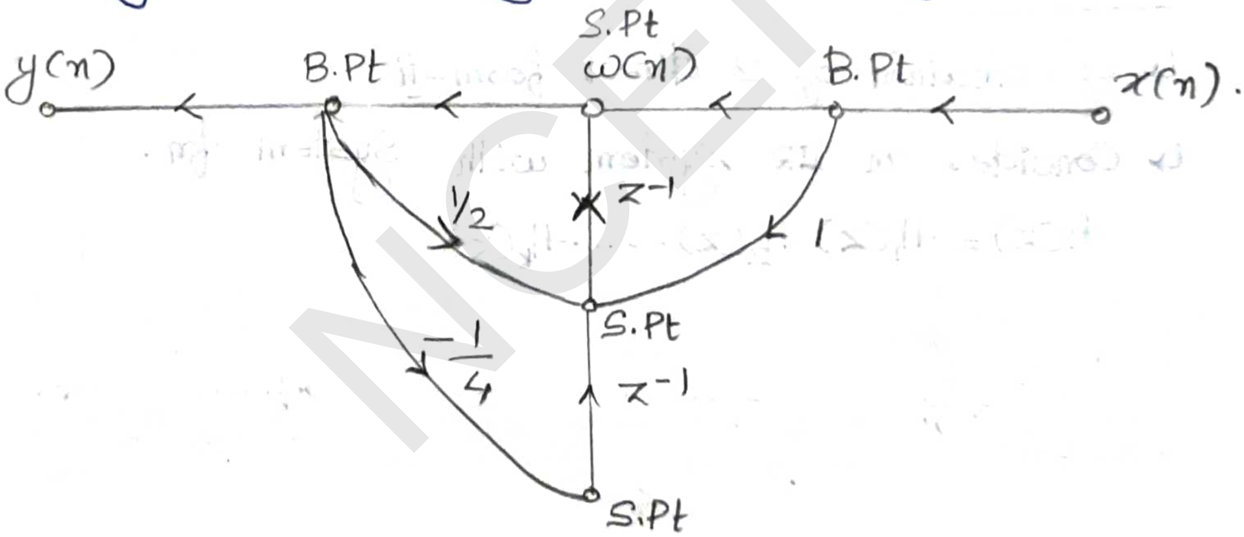
Signal flowgraph

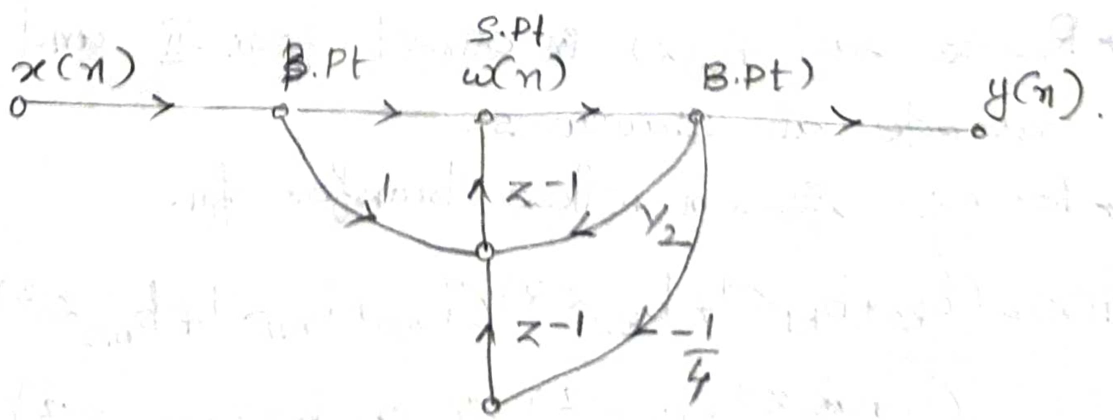
Transposed structure

(i) Change the direction of all branches.

(ii) Interchange the in/p and o/p.

(iv) change the summing pt to branching pt & Vice versa.





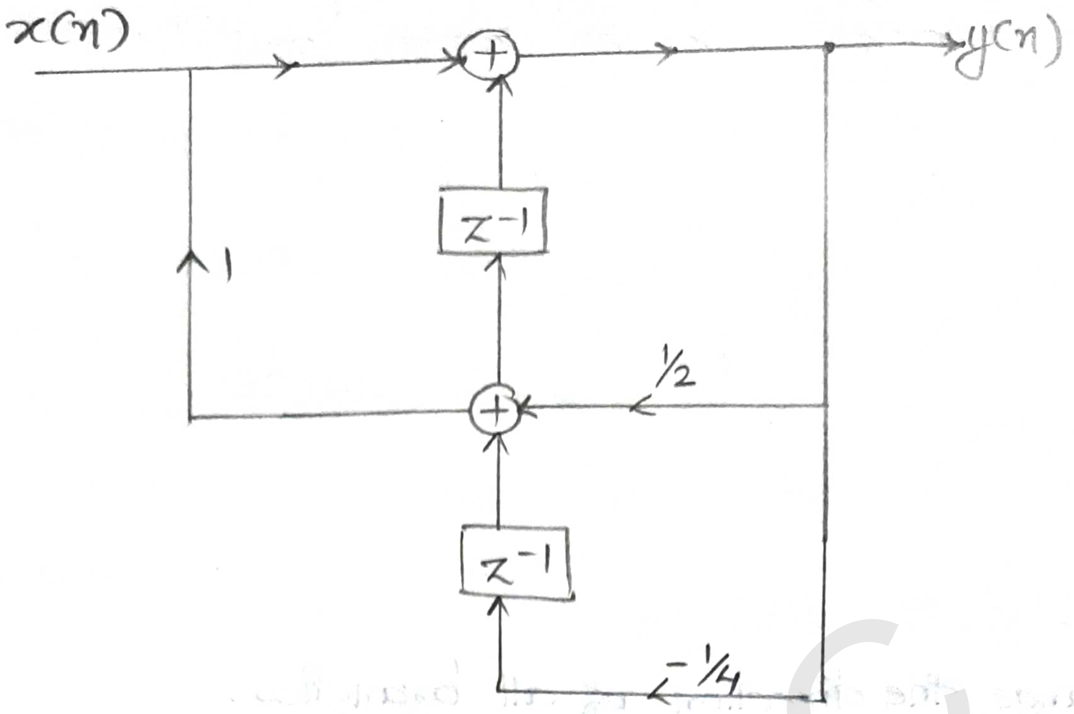
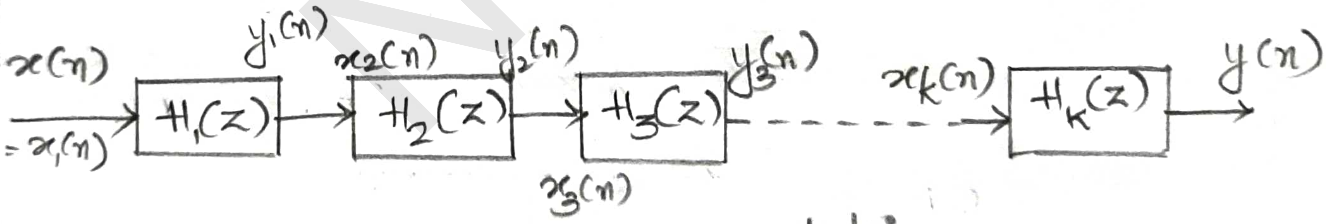


Fig :- Transposed Structure.

(iv) CASCADE FORM

- Combination of 2 direct form-II
- Consider an IIR system with system fn.

$$H(z) = H_1(z)\, H_2(z) \cdots H_k(z)$$



Block diagram representation.

- Realize each $H_k(z)$ in direct form-II and cascade all structures.
- For eg:- System whose transfer fn.

$$H(z) = \frac{(b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2})(b_{m0} + b_{m1}z^{-1} + b_{m2}z^{-2})}{(1 + a_{k1}z^{-1} + a_{k2}z^{-2})(1 + a_{m1}z^{-1} + a_{m2}z^{-2})}$$

$$H(z) = H_1(z)\, H_2(z).$$

$$H_1(z) = \frac{b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2}}{1 + a_{k1} z^{-1} + a_{k2} z^{-2}}$$

$$H_2(z) = \frac{b_{m0} + b_{m1} z^{-1} + b_{m2} z^{-2}}{1 + a_{m1} z^{-1} + a_{m2} z^{-2}}$$

→ Realizing $H_1(z)$ and $H_2(z)$ in direct form-II and cascading we obtain cascade form of the system fn.

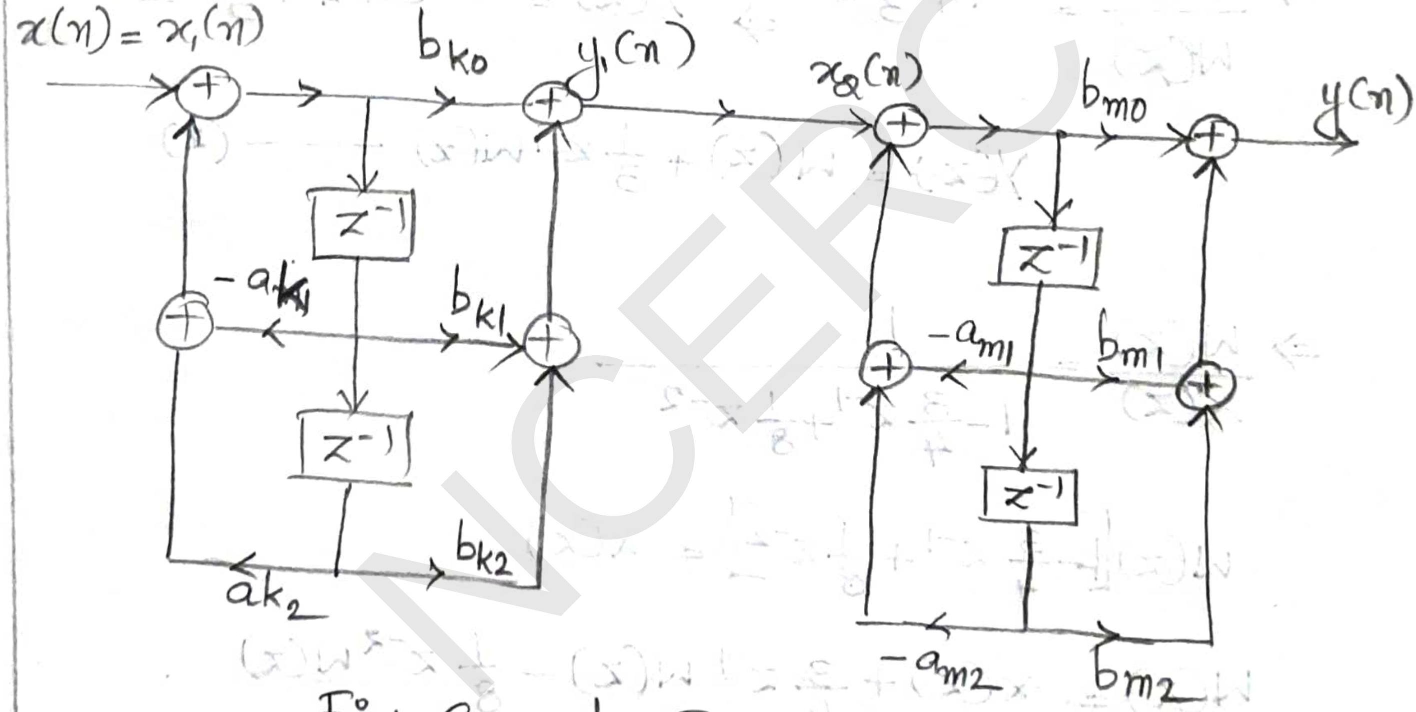


Fig: Cascade Realization

pbm

Qn. Realize the system with difference eqn.

$$y(n) = \frac{3}{4} y(n-1) - \frac{1}{8} y(n-2) + x(n) + \frac{1}{3} x(n-1)$$

in cascade form.

Soln.

$$y(n) = \frac{3}{4} y(n-1) - \frac{1}{8} y(n-2) + x(n) + \frac{1}{3} x(n-1)$$

$$Y(z) = \frac{3}{4} z^{-1} Y(z) - \frac{1}{8} z^{-2} Y(z) + X(z) + \frac{1}{3} z^{-1} X(z)$$

$$Y(z) - \frac{3}{4} z^{-1} Y(z) + \frac{1}{8} z^{-2} Y(z) = X(z) + \frac{1}{3} z^{-1} X(z)$$

$$Y(z)\left[1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\right]=X(z)\left[1+\frac{1}{3}z^{-1}\right]$$

$$\frac{Y(z)}{X(z)}=\frac{1+\frac{1}{3}z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} \quad \text{——— (1)}$$

$$\frac{Y(z)}{X(z)} = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$$

Try to make denominator as 2 fn.

Sum term $= \frac{-3}{4}$

Product term $= \frac{1}{8}$

$$= \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}-\frac{1}{4}z^{-1}+\frac{1}{8}z^{-2}}$$

$\frac{-1}{4} - \frac{1}{2} = \frac{-1-2}{4} = \frac{-3}{4}$

$$= \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}-\frac{1}{4}z^{-1}\left(1-\frac{1}{2}z^{-1}\right)}$$

$\frac{-1}{4} \times \frac{-1}{2} = \frac{1}{8}$

Factor $\Rightarrow \frac{-1}{4}, \frac{-1}{2}$

$$\frac{Y(z)}{X(z)} = \frac{1+\frac{1}{3}z^{-1}}{1(1-\frac{1}{2}z^{-1})-\frac{1}{4}z^{-1}(1-\frac{1}{2}z^{-1})}$$

$$H(z) = \frac{1+\frac{1}{3}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$$

$$H(z) = H_1(z) \cdot H_2(z)$$

$$H_1(z) = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}} \quad \text{and} \quad H_2(z) = \frac{1}{1-\frac{1}{4}z^{-1}}$$

$H_1(z)$ can be realized in direct form - II.

$$\frac{Y_1(z)}{X_1(z)} = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$\frac{Y_1(z)}{W_1(z)} \times \frac{W_1(z)}{X_1(z)}$$

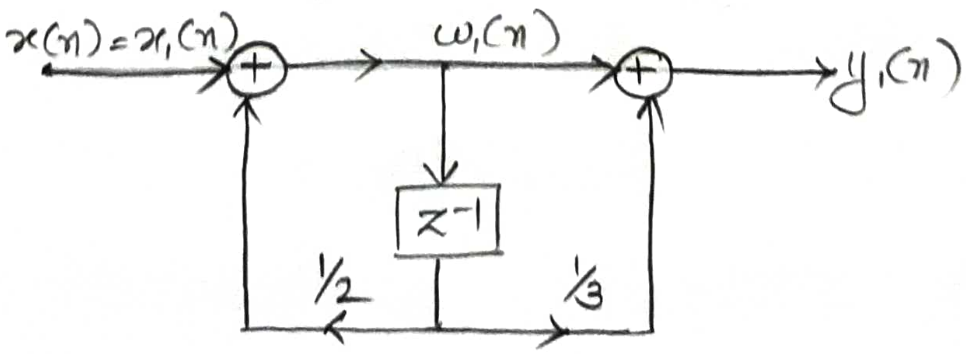
$$\Rightarrow \frac{Y_1(z)}{W_1(z)} = 1+\frac{1}{3}z^{-1} \Rightarrow Y_1(z) = W_1(z) + \frac{1}{3}z^{-1}W_1(z)$$

$$\Rightarrow \frac{W_1(z)}{X_1(z)} = \frac{1}{1-\frac{1}{2}z^{-1}} \Rightarrow W_1(z)\left[1-\frac{1}{2}z^{-1}\right] = X_1(z)$$

$$W_1(z) = X_1(z) + \frac{1}{2}z^{-1}W_1(z)$$

Taking Difference eqn.

$$y_1(n) = w_1(n) + \frac{1}{3}w(n-1) \quad \& \quad w_1(n) = x_1(n) + \frac{1}{2}w_1(n-1).$$



Similarly $H_2(z)$ can be realized in direct form II

$$H_2(z) = \frac{1}{1-\frac{1}{4}z^{-1}}$$

$$H_2(z) = \frac{Y_2(z)}{X_2(z)} \Rightarrow \frac{Y_2(z)}{W_2(z)} \times \frac{W_2(z)}{X_2(z)}$$

$$\Rightarrow \frac{Y_2(z)}{W_2(z)} = 1 \Rightarrow Y(z) = W_2(z)$$

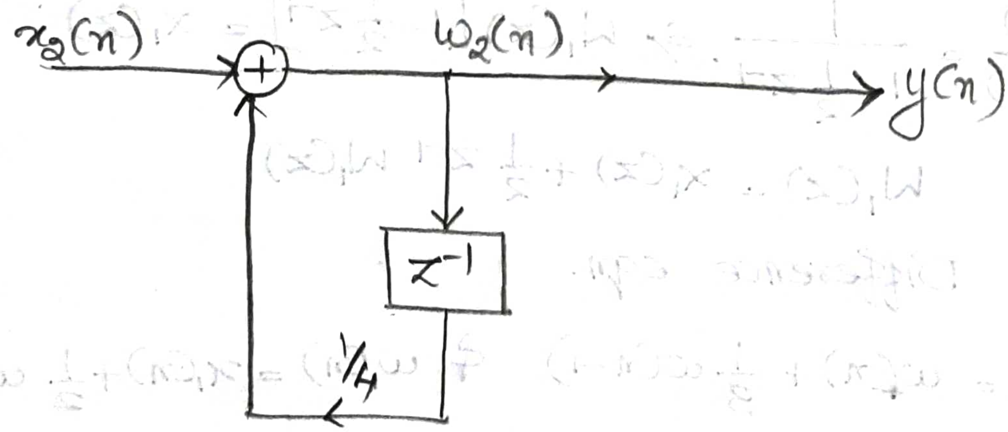
$$\Rightarrow \frac{W_2(z)}{X_2(z)} = \frac{1}{1-\frac{1}{4}z^{-1}} \Rightarrow W_2(z)\left[1-\frac{1}{4}z^{-1}\right] = X_2(z)$$

$$W_2(z) = X_2(z) + \frac{1}{4}z^{-1}W_2(z).$$

Taking difference eqn.

$$y(n) = \omega_2(n)$$

$$\omega_2(n) = x_2(n) + \frac{1}{4}\omega_2(n-1).$$



Cascading the realization of $H_1(z)$ and $H_2(z)$

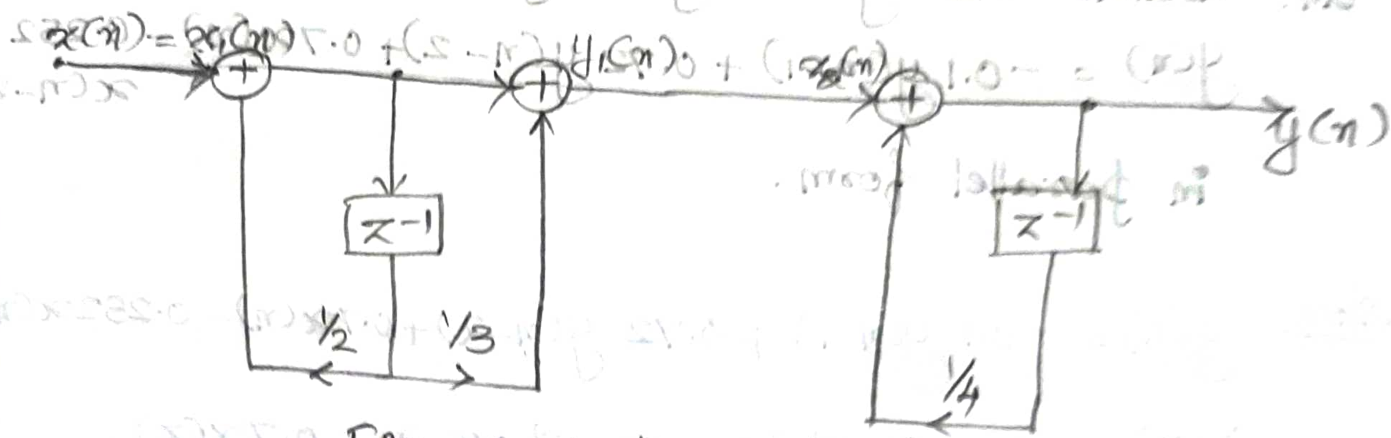


Fig:- Cascade Realization

(v) PARALLEL FORM STRUCTURE

↳ A parallel form realization of an IIR system can be obtained by performing a partial expansion of

$$H(z) = C + \sum_{k=1}^{N} \frac{C_k}{1-P_k z^{-1}}$$

where, $P_k \rightarrow$ Poles.

$H(z)$ can be written as

$$H(z) = C + \frac{C_1}{1-P_1 z^{-1}} + \frac{C_2}{1-P_2 z^{-1}} + \cdots + \frac{C_N}{1-P_N z^{-1}}$$

$$H(z) = \frac{Y(z)}{X(z)} = C + H_1(z) + H_2(z) + \cdots + H_N(z).$$

$$Y(z) = C\,X(z) + H_1(z)\,X(z) + H_2(z)\,X(z) + \cdots + H_N(z)\,X(z).$$

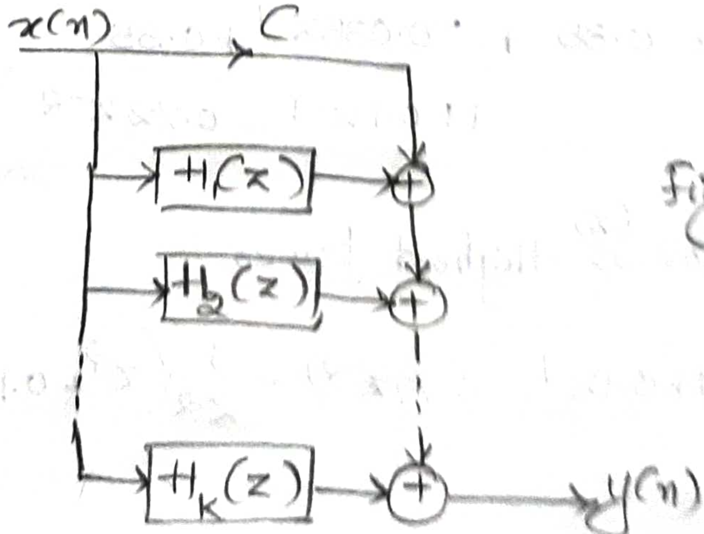


Fig:- Parallel form realization

Pbm.

Qn. Realize the system given by difference equation $y(n) = -0.1\,y(n-1) + 0.72\,y(n-2) + 0.7x(n) - 0.252\,x(n-2)$ in parallel form.

Soln.

$$y(n) = -0.1\,y(n-1) + 0.72\,y(n-2) + 0.7x(n) - 0.252x(n-2)$$

$$Y(z) = -0.1\,z^{-1}Y(z) + 0.72\,z^{-2}Y(z) + 0.7\,X(z) - 0.252\,z^{-2}X(z)$$

$$Y(z) + 0.1z^{-1}Y(z) - 0.72z^{-2}Y(z) = 0.7X(z) - 0.252z^{-2}X(z).$$

$$Y(z)\left[1 + 0.1z^{-1} - 0.72z^{-2}\right] = X(z)\left[0.7 - 0.252z^{-2}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{0.7 - 0.252z^{-2}}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

dividend. cascade form ← in C form.

→ Divisor

Divide ⇒ Long division method

$$\begin{array}{r} 0.35 \\ -0.72z^{-2} + 0.1z^{-1} + 1 \overline{\big) -0.252z^{-2} + 0.7} \\ -0.252z^{-2} + 0.035z^{-1} + 0.35 \\ \hline 0 - 0.035z^{-1} + 0.35 \end{array}$$

$$H(z) \Rightarrow 0.35 + \frac{-0.035z^{-1} + 0.35}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

Denominator $\overset{(x)}{\Rightarrow}$ Highest power

$$\frac{z^2}{z^2}\left(1 + 0.1z^{-1} - 0.72z^{-2}\right) = \frac{1}{z^2}\left(z^2 + 0.1z - 0.72\right).$$

$$= \frac{1}{z^2}\left[z^2 + 0.9z - 0.8z - 0.72\right]$$

Sum = 0.1
Product = 0.72
+0.9 −0.8

Rearranging

$$\frac{1}{z^2}\left(z(z+0.9) - 0.8(z+0.9)\right)$$

$$\frac{1}{z^2}\left[(z-0.8)(z+0.9)\right]$$

$$\frac{1}{z^2}\left[z(1-0.8z^{-1})\, z(1+0.9z^{-1})\right]$$

$$\Rightarrow (1-0.8z^{-1})(1+0.9z^{-1})$$

$$\therefore H(z) = 0.35 + \frac{0.35 - 0.035z^{-1} + 0.35}{(1-0.8z^{-1})(1+0.9z^{-1})}$$

Use partial fraction

$$\frac{-0.035z^{-1}+0.35}{(1-0.8z^{-1})(1+0.9z^{-1})} = \frac{A}{(1-0.8z^{-1})} + \frac{B}{(1+0.9z^{-1})}$$

$$0.35 - 0.035z^{-1} = A(1+0.9z^{-1}) + B(1-0.8z^{-1})$$

Put $z^{-1} = 0$

$$0.35 = A + B$$

Put $z^{-1} = 1$

$$0.35 - 0.035 = A(1.9) + B(0.2)$$

$$0.35 = A + B \quad \text{—— (1)}$$

$$0.315 = 1.9A + 0.2B \quad \text{—— (2)}$$

Solving eqn (1) and (2)

$$0.315 = 1.9A + 0.2B$$

$$(-) \quad 0.07 = 0.2A + 0.2B \quad \leftarrow \text{eqn (1)} \times 0.2$$

$$0.245 = 1.7A$$

$$A = \frac{0.245}{1.7} = 0.144 \qquad \boxed{A = 0.144}$$

Sub. A = 0.144 in eqn (1).

$$0.35 = 0.144 + B$$

$$B = 0.35 - 0.144$$

$$\boxed{B = 0.206}$$

$$H(z) = 0.35 + \frac{0.144}{1-0.8z^{-1}} + \frac{0.206}{1+0.9z^{-1}}$$

$$H(z) = C + H_1(z) + H_2(z).$$

$H_1(z)$ Realization

$$H_1(z) = \frac{Y_1(z)}{X_1(z)} \Rightarrow \frac{0.144}{1-0.8z^{-1}}$$

$$\frac{Y_1(z)}{W_1(z)} \times \frac{W_1(z)}{Y_1(z)}$$

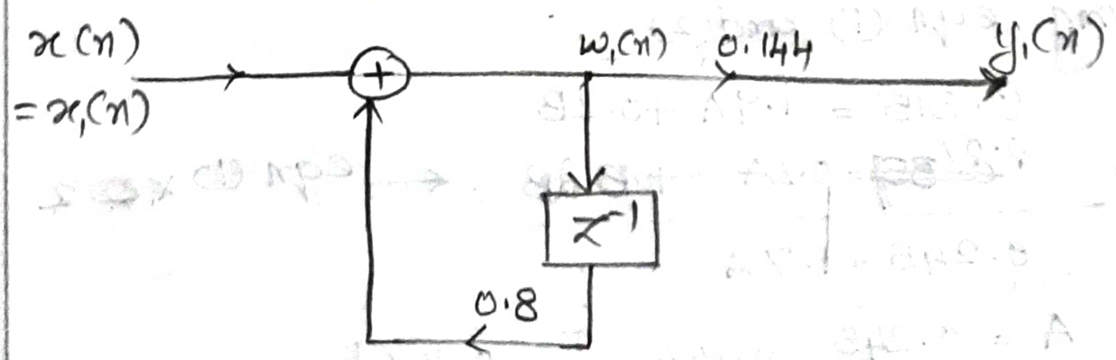
$$\frac{Y_1(z)}{W_1(z)} = 0.144 \Rightarrow Y_1(z) = 0.144\, W_1(z)$$

$$\frac{W_1(z)}{X_1(z)} = \frac{1}{1-0.8z^{-1}} \Rightarrow W_1(z) = X_1(z) + 0.8z^{-1}W_1(z)$$

Taking diff. eqn.

$$y_1(n) = 0.144\, \omega_1(n)$$

$$\omega_1(n) = x_1(n) + 0.8\, \omega_1(n-1).$$



$H_2(z)$ realization

$$H_2(z) = \frac{Y_2(z)}{X_2(z)}$$

$$\frac{Y_2(z)}{W_2(z)} \cdot \frac{W_2(z)}{X_2(z)} = \frac{0.206}{1+0.9z^{-1}}$$

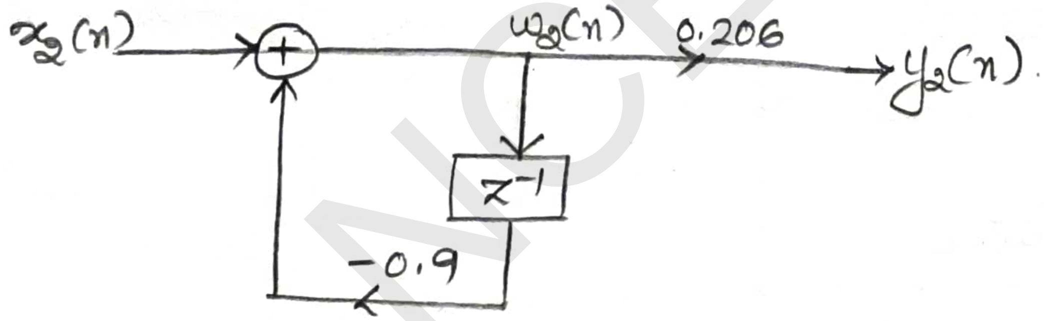
$$\frac{Y_2(z)}{W_2(z)} = 0.206 \Rightarrow Y_2(z) = 0.206\, W_2(z)$$

$$\frac{W_2(z)}{X_2(z)} = \frac{1}{1+0.9z^{-1}} \Rightarrow W_2(z) = X_2(z) - 0.9z^{-1}W_2(z)$$

Taking diff. eqn.

$$y_2(n) = 0.206\, w_2(n)$$

$$w_2(n) = x_2(n) - 0.9\, w_2(n-1).$$



Realization of H(z).

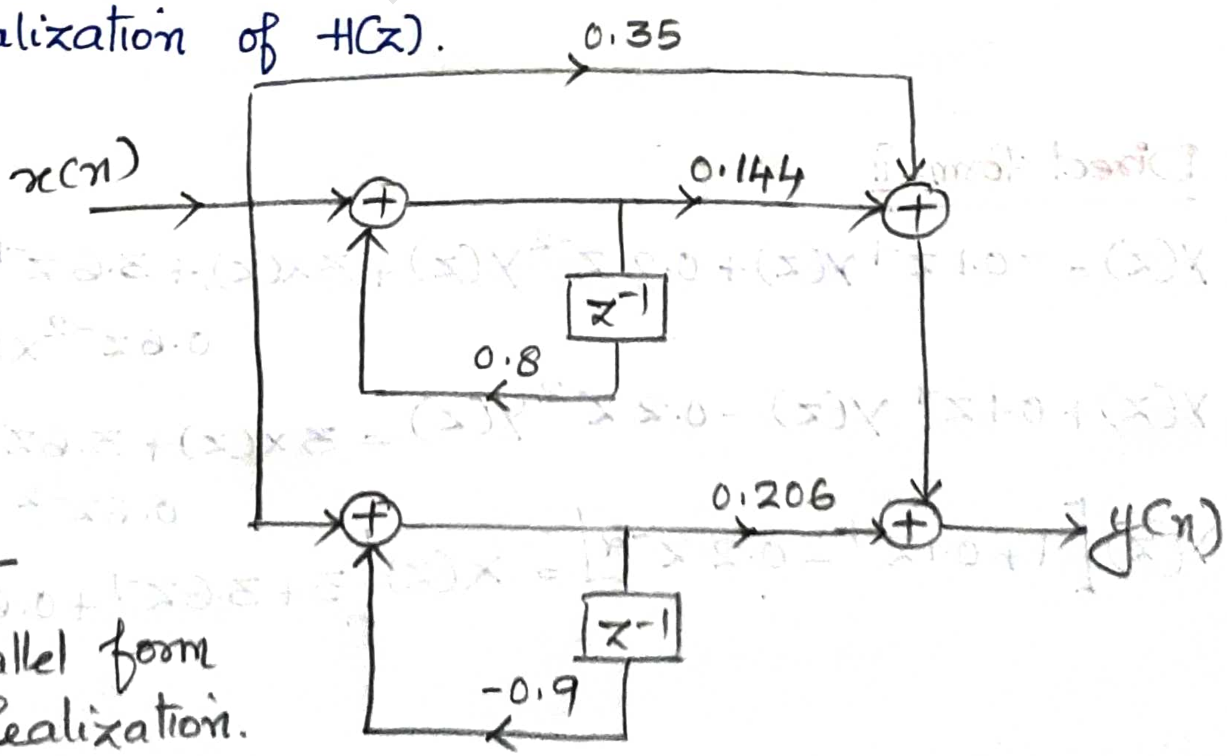


Fig:- Parallel form Realization.

MULTIRATE DIGITAL SIGNAL PROCESSING.

↳ Multirate DSP → in different rates → can analyze the signals.
can ↑e or ↓e the sample rate.

↳ The processing of a discrete time signal at different sampling rates in different parts of a system is called multirate DSP.

↳ Discrete time system that employ sampling rate conversion while processing the discrete time signals are called multirate DSP systems.

↳ Sampling rate conversion.
⇒ The process of converting a signal from one sampling rate to another sampling rate.

↳ There are 2 ways for sampling rate conversion in digital domain.

(1) Downsampling or Decemation
(2) Upsampling or Interpolation.

Down sampling or Decimation

The process of reducing the sampling rate by an integer factor 'D' or 'M'

Let $x(n)$ = discrete time signal

M or D → Sampling rate reduction factor.

$x(n) \rightarrow \boxed{\downarrow M} \rightarrow y(n) = x(Mn)$

$y(n) = x(Mn) \longrightarrow$ downsampled version of $x(n)$

↳ Down sampled signal $x(n)$ can be obtained by simply keeping every M^{th} sample and removing $(M-1)$ in between samples.

↳ Eg:- $x(n) = \{1, -1, 2, 4, 0, 3, 2\}$

(1) $y(n) = x(Mn)$ for $M=2$. (down sampling rate)

$M-1 = 2-1 =$

$M-1 = 1$

$x(n) = \{1, \cancel{-1}, 2, \cancel{4}, 0, \cancel{3}, 2\}$

$y(n) = \{1, 2, 0, 2\}$

(2) $M=3$ $\quad M-1 = 3-1 = 2$.

$x(n) = \{1, \cancel{-1}, \cancel{2}, 4, \cancel{0}, \cancel{3}, 2\}$

$y(n) = \{1, 4, 2\}$

discussed decimation in time domain.

Spectrum of down sampled signal

↳ decimation in frequency domain.

↳ $$Y(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j\frac{(\omega - 2\pi k)}{M}}\right)$$

↳ In Time domain → No. of samples ↓e → Freq. domain ↓ No. of Sample ↑e Stretch. (Expand)

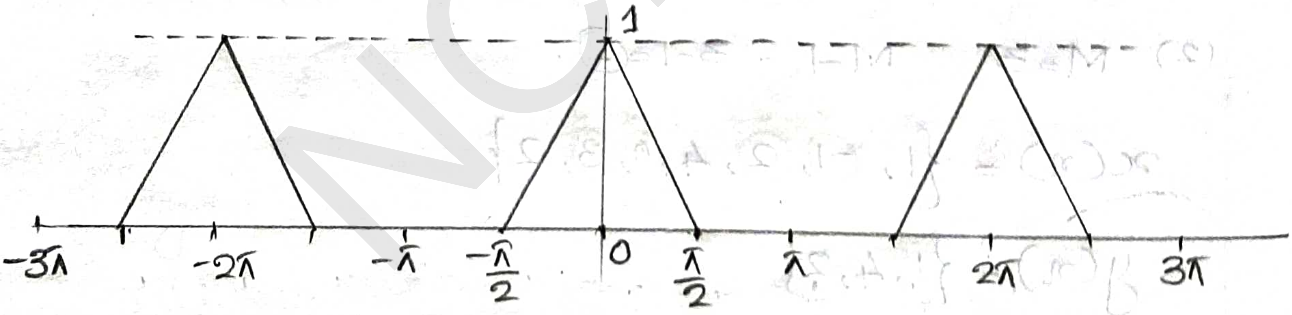
↳ Frequency domain → Compress ⟳ (Expand) Stretch. ← TD

↳ If the fourier transform of input signal of a down sampler is $X(e^{j\omega})$, then the FT of $Y(e^{j\omega})$ of the o/p signal $y(n)$

⇓

Sum of 'M' uniformly shifted and stretched versions of $X(e^{j\omega})$ scaled by a factor $\frac{1}{M}$.

Qn. Consider a spectrum of input signal $X(e^{j\omega})$ with a bandwidth of $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ shown below. When the signal is down sampled by a factor D, sketch the spectrum of a down sampled signal for sampling rate reduction factor D = 2, 3.



Soln. Case 1 D=2 or M=2.

$Y(e^{j\omega})$ = Spectrum of decimated signal.

$$Y(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j\left(\frac{\omega-2\pi k}{M}\right)}\right)$$

$$= \frac{1}{2}\sum_{k=0}^{1} X\left(e^{j\left(\frac{\omega-2\pi k}{2}\right)}\right)$$

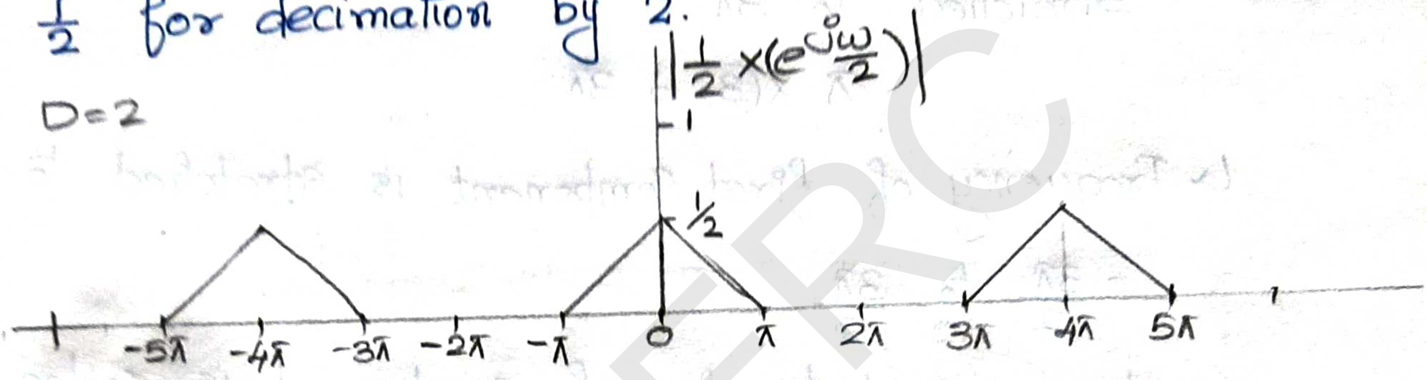
$$Y(e^{j\omega}) = \frac{1}{2}X\left(e^{\frac{j\omega}{2}}\right) + \frac{1}{2}X\left(e^{j\left(\frac{\omega-2\pi}{2}\right)}\right)$$

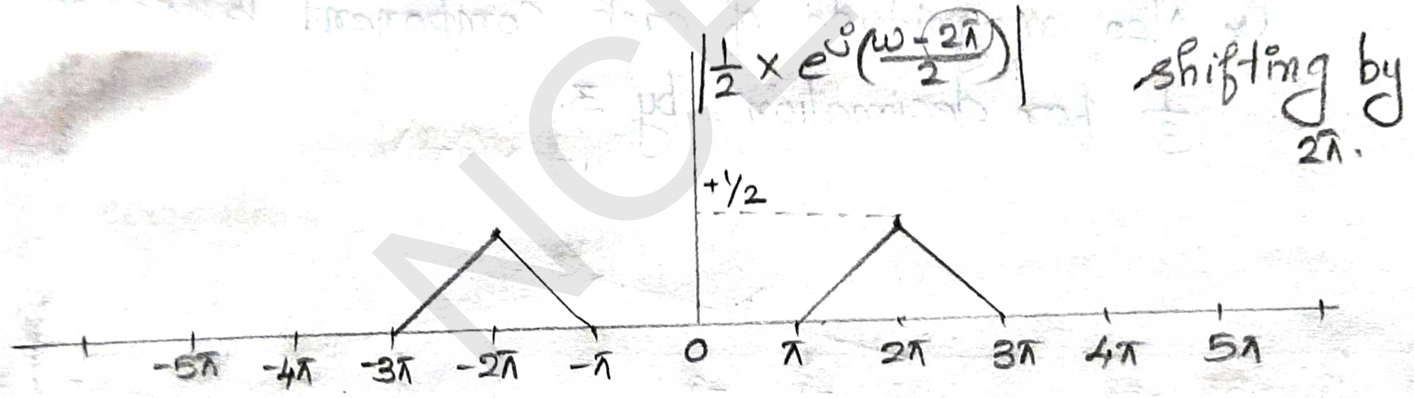
- Frequency range of i/p spectrum $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

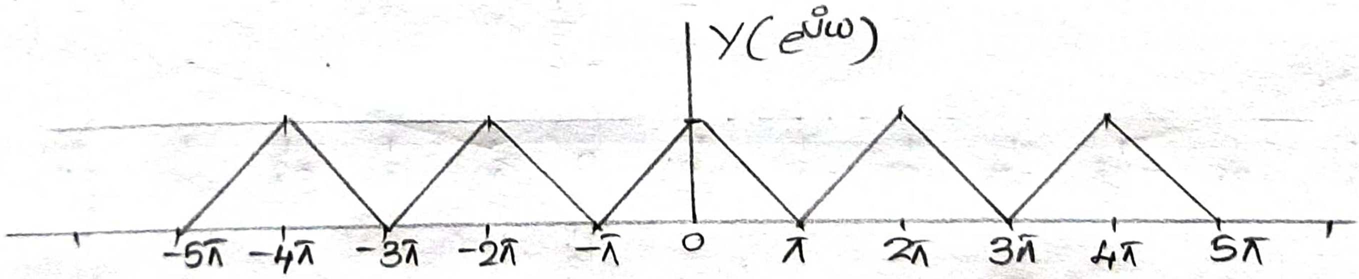
  Bandwidth (Upper − lower) $\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$

- When D=2 bandwidth is stretched to 2π.
- Frequency range of first component is stretched to $-\pi$ to $+\pi$.
- Also magnitude of each component scaled to $\frac{1}{2}$ for decimation by 2.

D=2







Case II , D=3

Let $Y(e^{j\omega})$ = Spectrum of decimated signal.

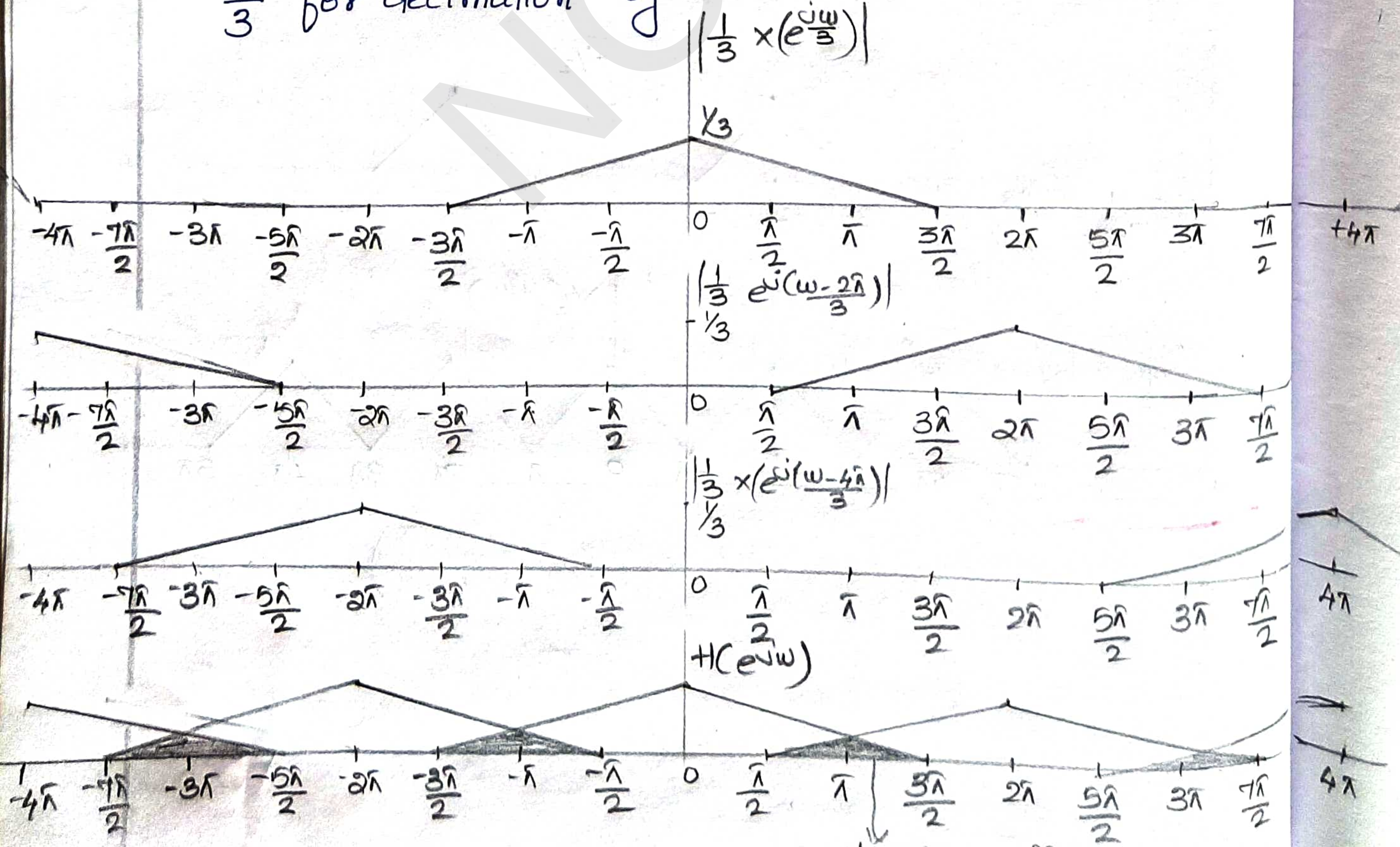
$$Y(e^{j\omega}) = \frac{1}{3}\sum_{k=0}^{2} X\left(e^{j\left(\frac{\omega-2\pi k}{3}\right)}\right)$$

$$= \frac{1}{3}X\left(e^{\frac{j\omega}{3}}\right) + \frac{1}{3}X\left(e^{j\left(\frac{\omega-2\pi}{3}\right)}\right) + \frac{1}{3}X\left(e^{j\left(\frac{\omega-4\pi}{3}\right)}\right)$$

- Bandwidth π of input spectrum is stretched to 3π

$$\pi \times ③ = 3\pi$$

- Frequency of first component is stretched to $-\frac{3\pi}{2}$ to $\frac{3\pi}{2}$
- Also magnitude of each component is scaled to $\frac{1}{3}$ for decimation by 3.



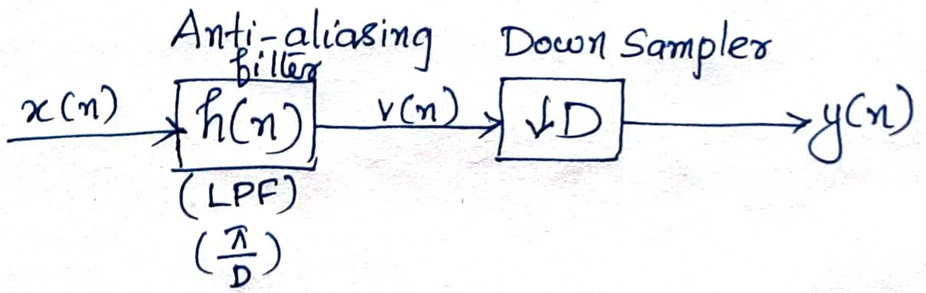
Aliasing effect.

Conclusion

From the above 2 cases of decimation, it is observed that for decimation by a factor D, as long as input spectrum is bandlimited to $\frac{\pi}{D}$ the spectrum of decimated signal does not overlap.

Anti-aliasing Filter

- When the input signal to decimation is not bandlimited then the spectrum of decimated signal has aliasing.
- Inorder to avoid aliasing the input signal should be bandlimited to $\frac{\pi}{D}$ for decimation by a factor D.
- Hence the o/p signal is passed through a 'LPF with a bandwidth of $\frac{\pi}{D}$ must be used before decimation.
- Since this LPF is designed to avoid aliasing in the o/p spectrum of decimation, it is called anti-aliasing filter.



Upsampling or Interpolation

- Interpolation ⇒ process of increasing the sampler of the discrete time signal.
- denoted as 'I' or 'L' → Sampling rate multiplication factor.
- Let $x(n)$ = discrete time signal.

$$y(n) = x\left(\frac{n}{I}\right) \text{ or } x\left(\frac{n}{L}\right)$$

upsampled version of $x(n)$

$$y(n) = \begin{cases} x\left(\frac{n}{L}\right) & n = 0, \pm L, \pm 2L \\ 0 & \text{otherwise} \end{cases}$$

- Upsampling the signal $x(n)$ by a factor L can be obtained by $L-1$ equally spaced zeros between each pair of samples.

Qn. Consider the discrete time signal $x(n) = \{1, 2, 3, 4\}$. Determine the unsampled version of the signals for the sampling rate multiplication factor $I = 2$, $I = 3$

$x(n) = \{1, 2, 3, 4\}$

$I = 2 \Rightarrow I - 1 = 2 - 1 = 1$

$I - 1 = 1$

$$y(n) = x\left(\frac{n}{2}\right) = \{1, 0, 2, 0, 3, 0, 4\}$$

$I = 3 \quad I - 1 = 2$

$$y(n) = x\left(\frac{n}{3}\right) = \{1, 0, 0, 2, 0, 0, 3, 0, 0, 4\}$$

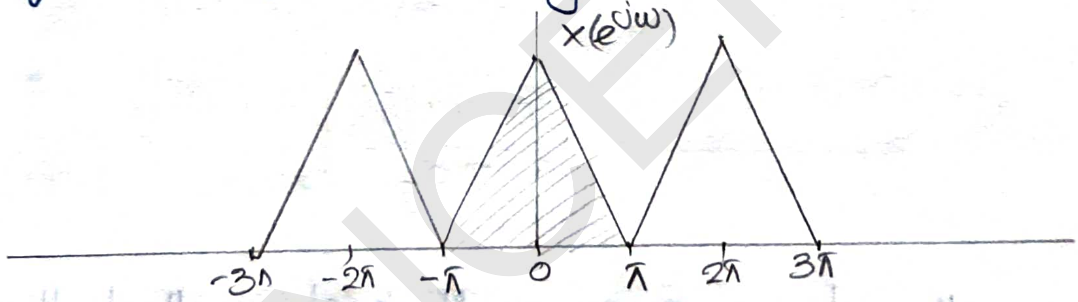
Spectrum of Upsampled Signal

$$Y(e^{j\omega}) = X(e^{j\omega I})$$

↳ The term $X(e^{j\omega I})$ is the frequency compressed version of $X(e^{j\omega})$ by a factor I.

↳ Since the frequency response is periodic with periodicity of 2π, the $X(e^{j\omega I})$ will repeat I times in a period of 2π in the spectrum of upsampled signal.

Qn. Eg: The spectrum of discrete time signal is shown below. Draw the spectrum of the signal if it is upsampled by $I = 2, 3$



Case I: $I = 2$

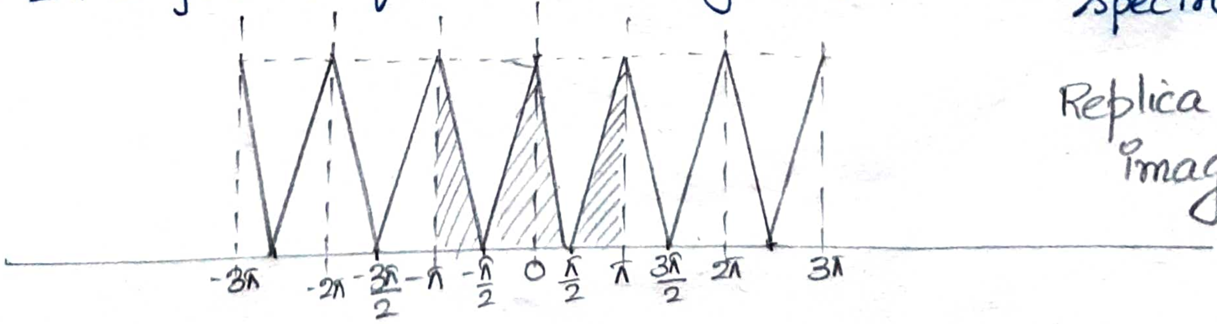
$Y(e^{j\omega}) = X(e^{j\omega I})$

$\Rightarrow Y(e^{j\omega}) = X(e^{j2\omega})$

Bandwidth of given Signal $= \pi - (-\pi) = 2\pi$

By $I=2 \rightarrow$ Bandwidth Compressed to π $\left[2\pi \times \frac{1}{2} = \pi\right]$.

∴ In a period of 2π – 2 images of Compressed spectrum.



Replica image

Case II

$I=3 \quad Y(e^{j\omega}) = X(e^{j3\omega})$

$$\text{Bandwidth} = 2\pi \times \frac{1}{3} = \frac{2\pi}{3}$$

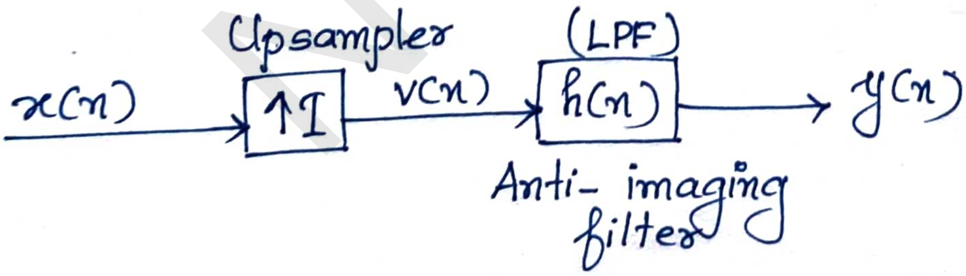
↳ Upsampled version of signal consists of 3 images of compressed spectrum in a period of 2π.

From the above 2 cases it is clear that the upsampled signal has multiple images in a period of 2π → depends upon the value of I.

↳ Time domain Expand → Frequency domain Compress.

Anti- Imaging Filter

- When upsampled by a factor of I, the output spectrum will have I images in a period of 2π, with each image bandlimited to $\frac{\pi}{I}$.
- Since this frequency spectrum in the range of 0 to $\frac{\pi}{I}$ are unique, need to filter the other images.
- Hence the o/p of upsampler is passed through a LPF with a bandwidth of $\frac{\pi}{I}$.
- Since this LPF is designed to avoid multiple images in o/p spectrum it is called anti-imaging filter.



EXAMPLE 5.2 Consider a signal x(n) = u(n).

(i) Obtain a signal with a decimation factor 3.

(ii) Obtain a signal with an interpolation factor 3.

Solution: Given that x(n) = u(n) is the unit step sequence and is defined as:

$$u(n) = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

The graphical representation of unit step sequence is shown in Figure 5.10(a).

(i) Signal with a decimation factor 3.
The decimated signal is given by

$$y(n) = x(Dn) = x(3n)$$

It is obtained by considering only every third sample of x(n). The output signal y(n) is shown in Figure 5.10(b).

(ii) Signal with interpolation factor 3.
The interpolated signal is given by

$$y(n) = x\left(\frac{n}{I}\right) = x\left(\frac{n}{3}\right)$$

The output signal y(n) is shown in Figure 5.10(c). It is obtained by inserting two zeros between two consecutive samples.

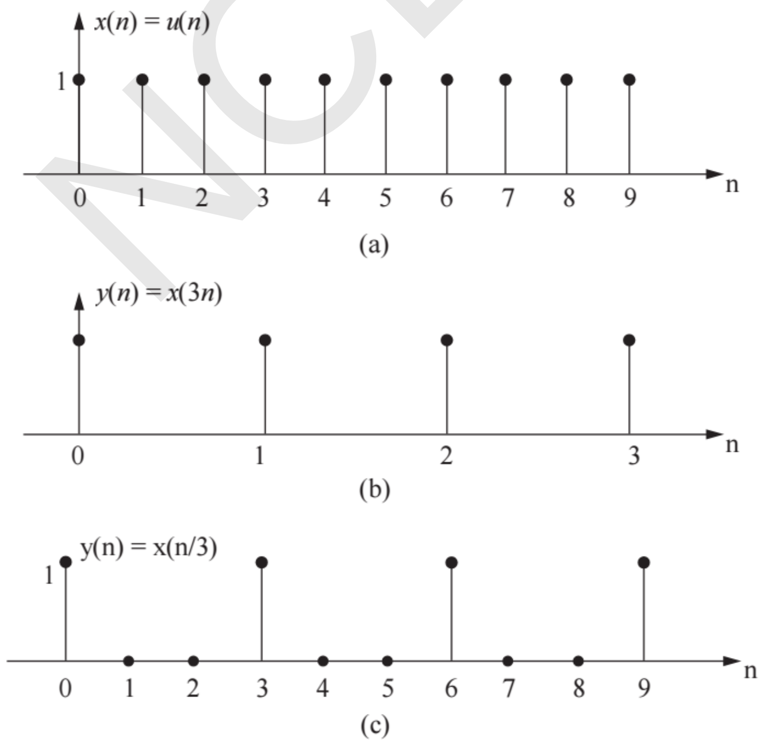


Figure 5.10 Plots of (a) x(n)=u(n), (b) x(3n) and (c) x(n/3).

EXAMPLE 5.3 Consider a ramp sequence and sketch its interpolated and decimated versions with a factor of 3.

Solution: The ramp sequence is denoted as r(n) and defined as

$$r(n) = \begin{cases} nu(n), & \text{for } n \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

The graphical representation of unit ramp signal is shown in Figure 5.11(a). The

decimated signal is given by

$$y(n) = r(Dn) = r(3n)$$

The output signal y(n) = r(3n) is shown in Figure 5.11(b). It is obtained by skipping 2 samples between every two successive sampling instants.

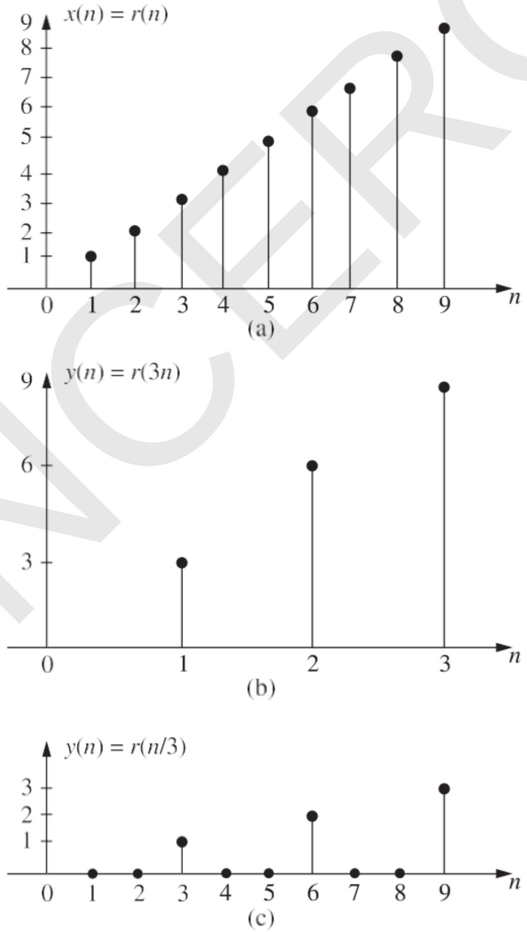


Figure 5.11 Plots of (a) r(n) = nu(n), (b) y(n) = r(3n) and (c) y(n) = r(n/3).

EXAMPLE 5.4 Consider a signal x(n) = sin nu(n).

(i) Obtain a signal with a decimation factor 2.

(ii) Obtain a signal with an interpolation factor 2.

Solution: The given signal is x(n) = sin n u(n). It is as shown in Figure 5.12(a).

(i) Signal with decimation factor 2. The signal x(n) with a decimation factor 2 is given

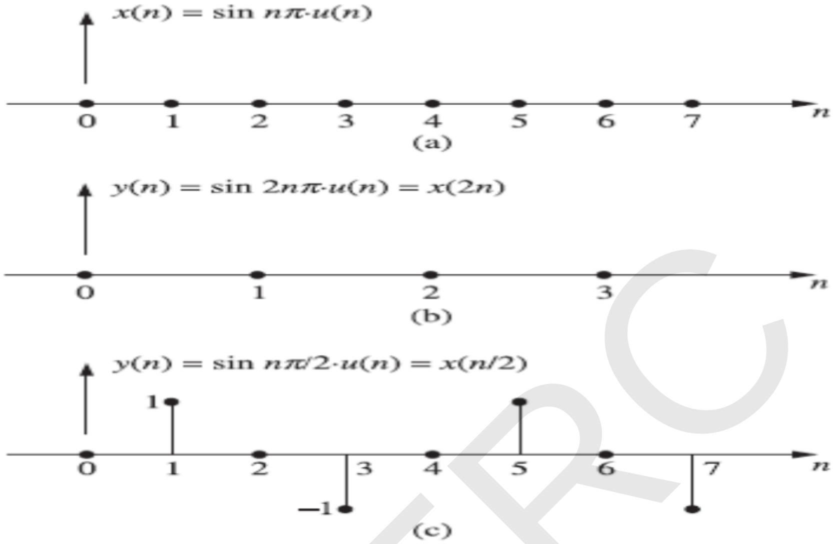


Figure 5.12 Plots of (a) x(n) = sin n u(n), (b) y(n) = sin 2n u(n) and (c) y(n) = sin (n /2)u(n).

MODULE - V

Computer architeture for signal processing.

↳ Programmable digital signal processors (PDSPs)
→ general purpose microprocessors designed specifically for DSP applications.

↳ They contain special architecture and instruction set so as to execute computation - intensive DSP algorithms more efficiently.

↳ Programmable DSPs can be divided into two categories.

(i) General purpose digital signal processor.

↳ These are basically high speed microprocessors with architecture and instruction sets optimized for DSP operations.

Eg:- Fixed point processors ⇒ Texas Instruments TMS320C5X, TMS320C54X
MOTOROLA DSP 563x

Floating point processors ⇒ Texas instruments TMS320C4X, TMS320C67xX,
Analog devices ADSP21xxx

(ii) Special purpose digital signal processors.

↳ These type of processors consist of hardware

* designed for specific DSP algorithms such as FFT
* hardware designed for specific applications such as PCM and filtering.

Eg:- FFT processor PDSP(16515A)
Programmable FIR filter _ UPDSP 16256

↳ 1980s → No. of PDSPs appeared in the commercial markets.

↳ 1979 → Intel introduced first digital signal processor - (Intel 2920)

↳ Texas instrument introduced TMS32010 → First generation fixed point DSP
↳ NMOS technology.

→ Later TMS320C20 introduced
↳ CMOS technology.

↳ Factors that influence the selection of a DSP processor for a gn. application
* Architectural features
* Execution speed
* Type of arithmetic
* Word length.

↳ Applications of PDSPs.
* Communication systems.
* Audio signal processing.
* Control and data acquisition
* Biometric information processing
* Image/video processing...

Von Neumann Architecture

- 1946 → John Von Neuman developed first computer architecture that allowed the computer to be programmed by codes residing in memory.
- Instructions were stored in ROM.
- Von Neumann architecture widely used in majority of microprocessors.
- in Von Neumann Arch^{r}.

CPU.
(either)

Read an instruction (OR) Reading/writing data from/to the memory.

- Both can't occur at the same time → Since the instruction and data use the same signal pathways and memory.
- Consists of 3 buses.
  - (i) Data bus ⇒ Transports data b/w CPU and its peripherels
    - bidirectional.
    - CPU can read or write data in the peripherals.
  - (ii) Address Bus ⇒ CPU uses address bus to indicate which peripherals it want to access
    - within each peripheral which specific register.
    - Unidirectional.
    - CPU always write the address, which is ready by the peripherals.

(iii) Control bus ⇒ The bus carries signals that are used to manage and synchronize the exchanges between the CPU and its peripherals.

↳ indicates if the CPU wants to read or write the peripheral.

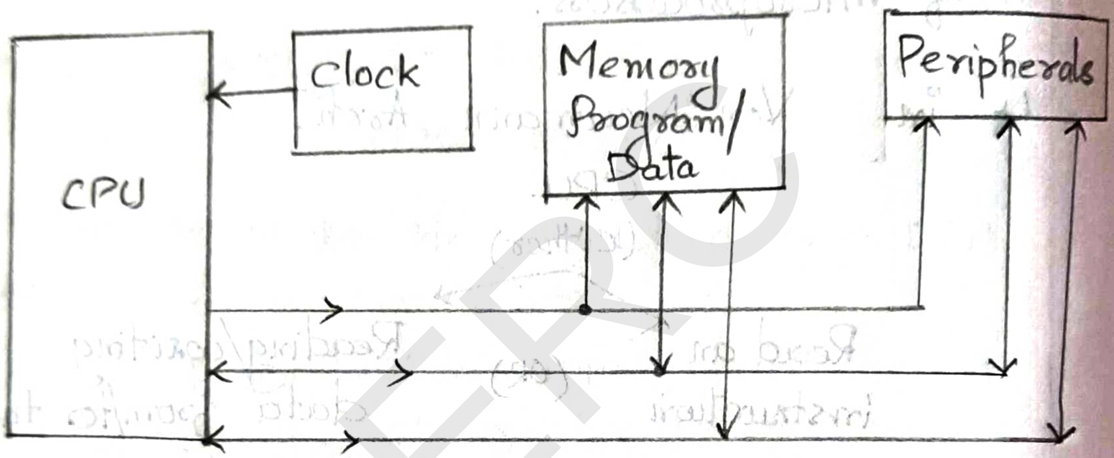


Fig:- Von Neumann Architecture.

↳ Main characteristics

* it only possesses 1 bus system.
* Same bus carries all the information exchanged b/w CPU and the peripherals including the instruction codes as well as data processed by the CPU.

Harvard Architecture

↳ Term Harvard originated from the Harvard mark-1 relay-based computer → stored instructions on punched taped and data in relay latches.

↳ Harvard Archⁿ ⇒ physically separates memories for their instructions and data → requiring dedicated buses for each of them.

↳ Instructions and operands can be fetched → simultaneously.

↳ Most DSP Processors → use modified H. Archⁿ. → with 2 or ~~mor~~ 3 memory buses.
  ↳ allowing access to filter coefficients and i/p signals in the same cycle.

↳ it possess two independent bus systems

↳ capable of simultaneous reading an instruction code and reading/writing a memory or peripheral as a part of the execution of previous instruction.

↳ Two memory ⇒ not possible for the CPU to mistakenly write codes into the program memory → Compute the code while it is executing.

↳ Less flexible

↳ needs two independent memory banks

↳ These two resources are not interchangeable.

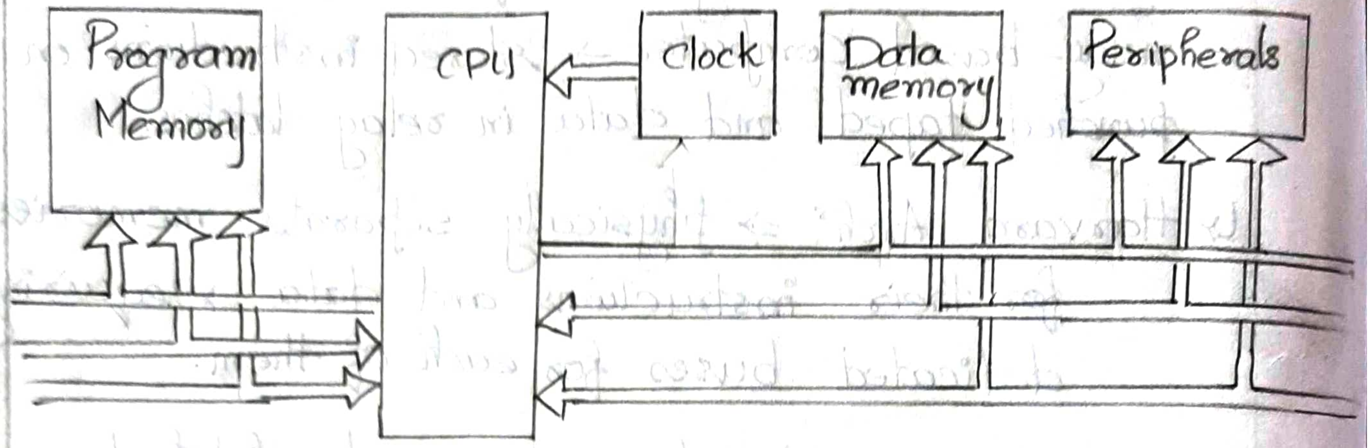


Fig :- Harvard Architecture

- Modified Harvard Archr → Used DSP's multiport memory.
  → has separate bus systems for program memory and data memory and in/op peripherals.
- have multiple bus system for program memory alone or for data memory alone.
- These multiple bus system increases complexity of the CPU → but allow it to access several memory locations simultaneously → there by increasing the data throughput b/w memory and CPU.

Multiply Accumulate Unit (MAC)

- MAC operation ⇒ basis of many digital signal processing algorithms → notably digital filtering.
- Digital filter ⇒ Refers to an algorithm by which a digital signal or sequence of numbers is transformed into another sequence of numbers termed the o/p digital signal.

Digital Signals / Seq. of no.s —— Transform ——→ digital o/p signal or Another sequence.

- Digital filters ⇒ signals in digital domain (discrete-time signals).
- Used extensively in applications such as
  - * DIP → Digital image processing
  - * Pattern recognition
  - * Spectral analysis.
- FIR filters preferred in lower order solutions
- they do not employ feedback → they exhibit naturally bounded response.
- Simpler to implement
- Require one RAM location, and one co-efficient for each other.
- O/p of the FIR filter is given by

$$y(n) = \sum_{k=0}^{N-1} x(n)\, h(n-k).$$

$x(n) \rightarrow$ i/p to the filter $y(n) \rightarrow$ o/p of the filter.

$h(n-k)$ impulse response

↳ o/p of the filter ⇒ a finite length weighted sum of the present and past i/ps to the filter.

↳ To perform filtering through eqn. → minimum requirement is to quickly multiply 2 values and add the result.

↳ To making it possible → fast dedicated hardware MAC using either fixed point or floating Point arithematic is mandatory.

↳ characteristics.

* 16×16 bit 2's Complement i/ps
* 16×16 bit multiplier with 32-bit product in 25ns.
* 32/40 bit accumulator.

↳ Eg:- TMS320C50 → FIR eqn can be ~~effectivel~~ efficiently implemented using the instruction pair.

RPT NM1 ——→ Loads (N-1) in repeat instruction Counter.

MACD HNM1, XNM1 ---→ MAC with data move instruction following it to be repeated N times.

↳ MACD instruction performs a no. of operations in one cycle.

(1) Multiplies the data sample $h(n-k)$ → in the data memory by the coefficient $x(n)$ in the program memory.

(2) Adds previous product to the accumulator.

(3) Implements the unit delay, symbolized by z^{-1} by shifting the data sample $h(n-k)$ upto update the tapped delay time.

MAC Function

- MAC speed applies both to FIR and IIR Filters.
- MAC step performs the following.
  * Reads a 16-bit sample data (pointed by a register)
  * Increments the sample data pointer by 2
  * Reads a 16 bit coefficient
  * Increments the coefficient register pointer by 2
  * Sign multiply (16-bit) data and co-efficient to yield a 32-bit result.
- TMS320C54x MAC unit performs 16x16 → 32-bit fractional MAC operation in a single instruction cycle.
- Multiplier supports → Signed/Signed multiplication
  Signed/Unsigned "
  Unsigned/Unsigned "
- These operations allow efficient extended-precision arithmetic
- Many instructions using the MAC Unit can optionally specify automatic round-to nearest rounding.

Pipelining

- Most of the early μPs execute instructions → sequentially.
- After the ~~first~~ execution of first instruction → next one starts.
- Problem ⇒ extremely inefficient
  - 2nd instruction has to wait untill all the steps of 1st instruction are completed.
- To improve the efficiency → advanced microprocessors and digital signal processors → Use an approach called pipelining
- Pipelining ⇒ different phases of operation and execution of instructions are carried out in parallel.
- In modern processors ⇒ First step of execution is performed on the first execution → instruction passes to the next step → a new instruction is started.
- The steps in the pipeline are often called stages
- Basic action of any μP ⇒ 4 Simple steps.

  (i) The Fetch phase (F) ⇒ Next instruction is fetched from the address stored in the program Counter.

  (ii) The decode phase (D) ⇒ Instruction in the instruction register is decoded and the address in the program Counter is incremented

  (iii) Memory Read (R) phase ⇒ reads the data from the data ~~buses~~ buses and also writes data to the data buses.

(iv) The execute phase (X) ⇒ executes the instruction currently in the instruction register and also complete the write process.

↳ In modern processor → above 4 steps get repeated over and over again until the program is finished executing.

↳ Each of above stages ⇒ represent one phase in the "lifecycle" of an instruction.

↳ Instruction starts → from fetch phase → decode phase → Memory read phase → finally to execute phase

↳ each phase take fixed and equal amount of time.

↳ Pipelining a processor ⇒ breaking down ~~&~~ its instruction into a series of discrete pipeline stages → can be completed in sequence by specialized hardware.

↳ Instruction's Lifecycle ⇒ Four phases.

↳ Instruction execution process divided into a sequence of 4 discrete pipeline stages.

↳ Each pipeline stage corresponds to a phase in the standard instruction ~~cy~~ lifecycle.

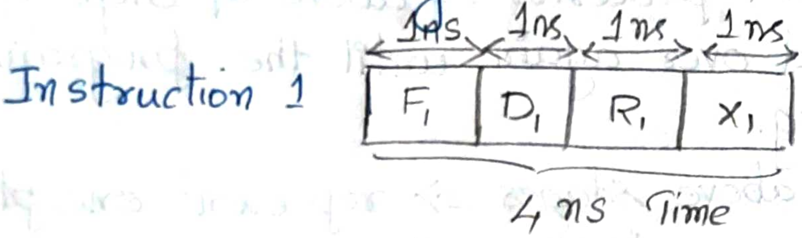
↳ No. of pipeline stages ⇒ pipeline depth.

∴ Four-stage pipeline ⇒ pipeline depth of four.

↳ Lets assume that No. of stages = 4 & execution time of an instruction = 4 ns.

↳ Time taken for each stage in the instruction is equal → then time taken for each stage is 1 ns.

↳ Original single cycle processor's ⇒ 4 ns → broken down into 4 discrete - sequential pipeline stages of 1 ns each in length.



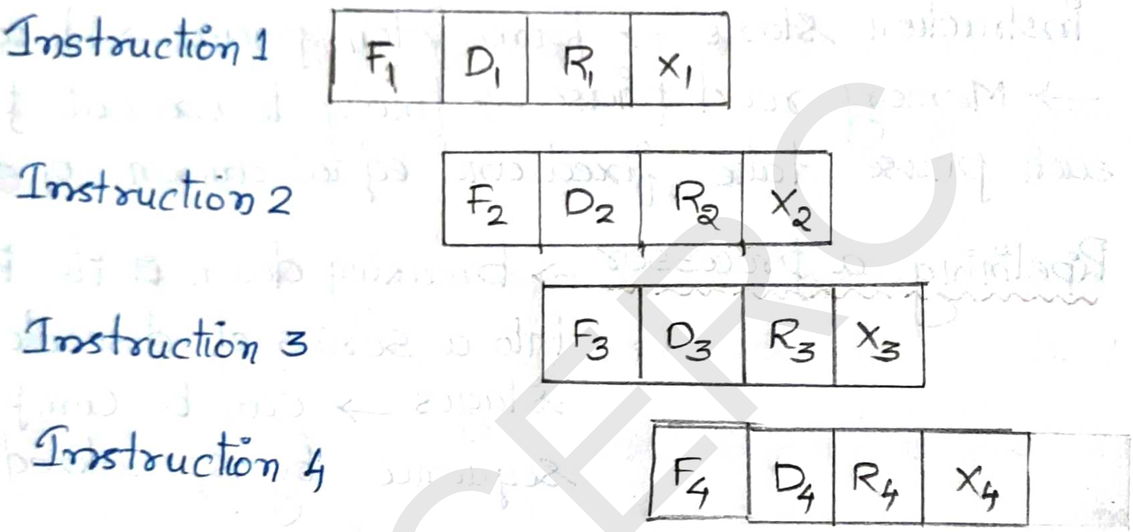


Fig:- Four stages of TMS320C54x

↳ At beginning → ~~First instruction~~

First ns → First instruction enters the Fetch stage.

↳ 1ns complete → 2nd ns begins → First instruction moves to Decode stage.

→ 2nd Instruction enters the fetch stage.

↳ 3rd ns begins → Instruction 1 moves to R_1 (memory stage)

→ Instruction 2 moves to Decode stage (D_2)

→ Instruction 3 moves to Fetch stage (F_3)

↳ 4th ns begins → Instruction 1 moves to execution stage (X_1)

→ Instruction 2 → moves to Memory stage (R_2)

Instruction 3 → moves to Decode stage (D_3)

Instruction 4 → moves to Fetch stage (F_4)

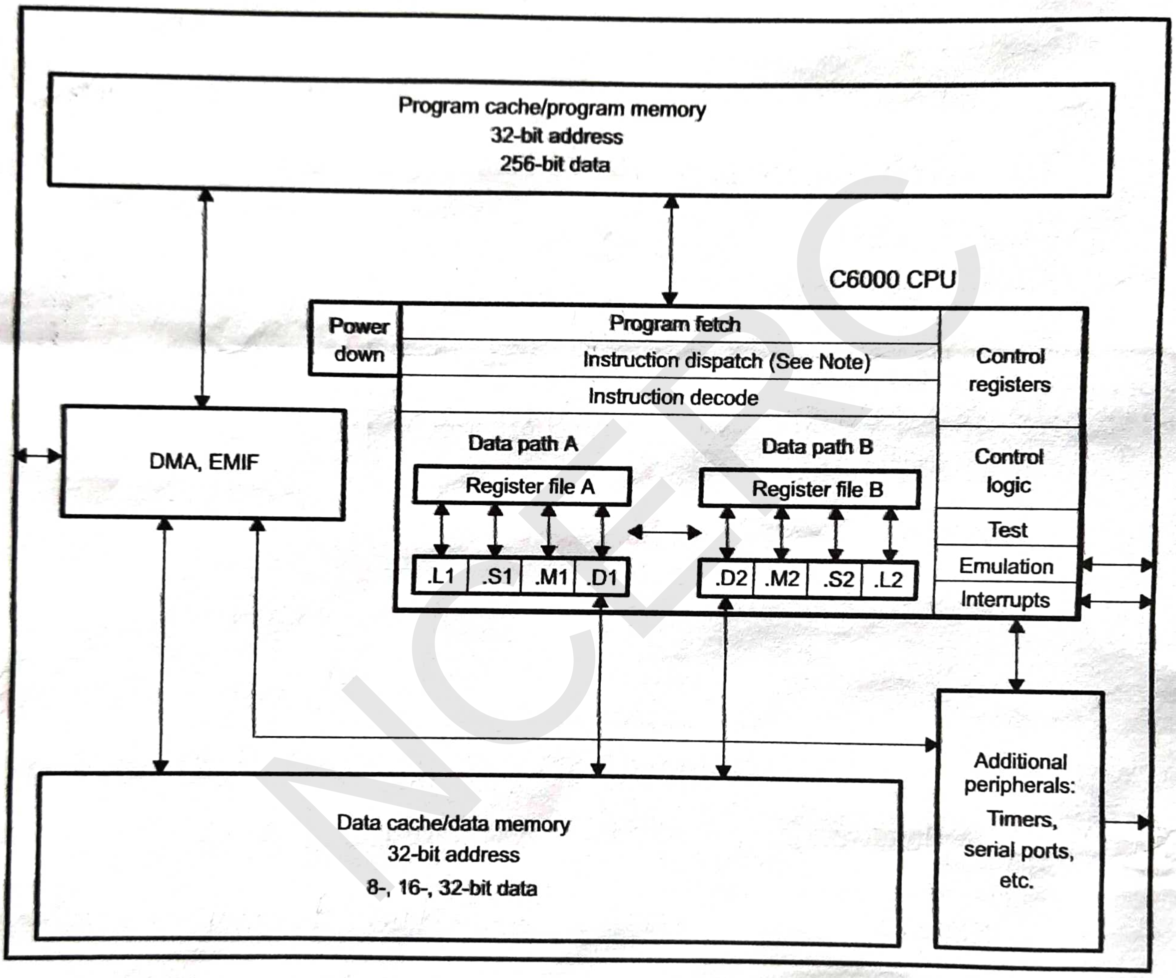
- After 4ns → fully elapsed → 5th ns starts → Instruction 1 passed from timeline → finished executing.
- End of 4ns → pipelined processor has completed one instruction.
- At the start of 5th ns → pipeline is now full (Instruction 2)
- Processors can begin completing instructions at a rate of one instruction per nanosecond.
- 4 instruction => 16 nanoseconds.
- Pipeline in different TMS320 processors.

DSP Processors	Pipeline phases
TMS320C2000	F-D-R-X (4 levels)
TMS320C3x	F-D-R-X (4 levels)
TMS320C5x	F-D-R-X (4 levels)
TMS320C54x	PF-F-D-A-R-X (6 levels)

- TM320C54X → Two additional phases.
  Prefetch (PF) stores the address of the instruction to be fetched.
  Access phase (A) → reads the address of the operand and modify the auxiliary register and stack pointer (if required)
- Pipelining => dramatic improvements in system performance.

TMS320C67xx ARCHITECTURE

Figure 1–1. TMS320C67x DSP Block Diagram



- TMS320C67xx → Digital Signal processor
- from Texas instrument
- Used for processing of digital signals.
- Consists of program memory, data memory, CPU, Additional Peripherals and DMA, EMIF.
- C6000 CPU device → Comes with program memory
  → can be used as program cache.
- The devices also have varying sizes of data memory.
- DMA – Direct memory access
  EMIF – External memory interface
  Power down logic
  } Usually come with the CPU
- Peripherals such as Serial ports and host ports } only on certain devices.

CPU

- Consists of Program fetch, Instruction dispatch, Instruction decode
- First fetch the instructions is to be processed → dispatch the instructions to various units → decode → process
- From program memory → all the instructions are taken
- From data memory → all the required data will be taken
- 2 Data paths ⇒ Data path A and Data Path B
- Data path ⇒ processing Unit
- Data Path A consists of Register file A and
  Data Path B ⇒ Register file B.

↳ Each of the register file consists of 16 register files

Register File A ⇒ 16 Registers } 32 Regr.
Register File B → 16 Regrs. } ↳ 32 bit size

↳ .L, .S, .M and .D ⇒ Functional Units

↳ .L_1, .L_2, .S_1, .S_2, .D_1 and D_2 ⇒ ALU operation

↳ ~~D1 and D2~~ ⇒

.M_1 and .M_2 ⇒ Multiply operations

L → Logic operation
S → Store "
M → Multiplication
D → Data transfer

↳ 6 ALUs and 2 - MUL

↳ A Control regr. file provides the means to Configure and control various processor operation.

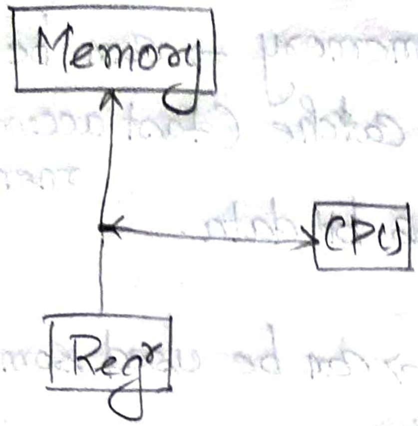
↳ ~~No~~ After all the steps, → now CPU have decoded instruction → Collect the data from data memory and decode it.

↳ Controlling and co-ordination of these operations are done with the help of Control regr, Control logic.

↳ Power down Unit → Reduce the power consumption of DSP.

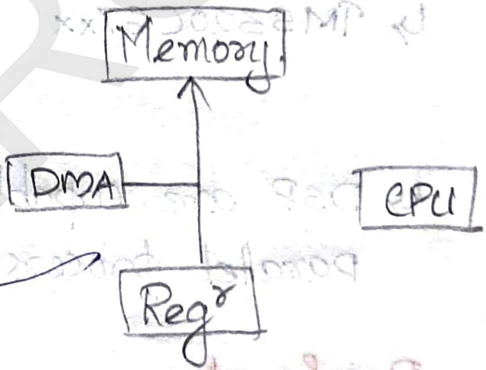
* DSP works on clock cycle
* Consumes more power
* Power down unit → Reduce the clock rate
→ Reduce the power Consumption.
* DSP in power saving mode → Power down Unit
↳ ACTIVE

DMA, EMIF



- Reg^r ~~wants to~~ needs to access some data from memory
- Reg^r can't access the data directly.
- Reg^r has to ask permission from CPU → CPU will grant the permission to access Reg^r for accessing (Read/Write) the data from memory.

- if DMA is present
- DMA ⇒ Direct memory access
- Reg^r access the memory directly without seeking the permission from CPU



- "Without the influence of CPU → Reg^r can directly access the memory." → Accessing is controlled by DMA controller.

- EMIF ⇒ External Memory Interfacing
- In addition to internal memory → we can also add some external memory to DSP
- SD RAM, S RAM, SB RAM, & various memories can be added.
- During EMIF → Program memory & Data memory kept as separate → will be unified together.

Memory

↳ Program Catche / Program memory ⇒ can be used sometimes as catche (Fast accessing memory).

* 32 bit address → 256 bit data.

↳ Data cache/Data memory ⇒ can be used sometimes as cache

* 32 bit address
* 8, 16, 32 bit data

↳ TMS320C67xx follows VLIW architecture (Very Large Instruction Word)

↳ DSP are capable of Processing VLIW → allows parallel processing of these instructions.

Peripherals

↳ DMA Controller → transfers data b/w data memory to program memory without intervention by the CPU.

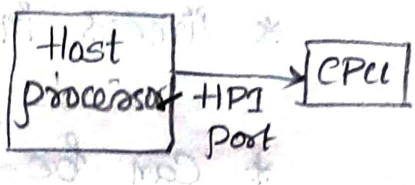
* DMA Controller has 4 programmable channels
* 1 auxiliary channel (5th channel)
* 4 + 1 ⇒ 5 channels.

↳ EDMA Controller → (Extended DMA)

* performs the same fn as the DMA Controller.
* 16 programmable channels.
* RAM Space to hold multiple configurations for future transfers.

↳ HPI (Host-port Interfacing)

* Parallel port



* Using HPI port → host processor can directly access CPU's memory space.
* Host and CPU can exchange information via internal or external memory.
* The host has direct access to memory-mapped peripherals.

↳ Expansion Bus

* Replacement of HPI
* Expansion of EMIF
* Expansion provides two distinct areas of functionality (host port and I/O Port) which can co-exist in a system.
* Host port of the expansion bus → operate either in (i) asynchronous slave mode
  (ii) synchronous master/slave mode.
* This allows the device to interface to a variety of host bus protocols.

↳ McBSP (Multichannel buffered serial port)

* Based on standard serial port interface
* Port can buffer serial samples in memory automatically with the aid of the DMA/EDMA Controller.

↳ Timers

* Timers in C67xx ⇒ Two 32 bit general purpose timers.

* Can be used for the following functions.
  - (i) Time events
  - (ii) Count events
  - (iii) Generate Pulses
  - (iv) Interrupt the CPU
  - (v) Send synchronization events to the DMA/EDMA controller.

↳ Additional Features.

* Hardware of the DSP Processor → Supports single precision and double precision format.
  (32 bit Operations) (64-bit floating pt operations)

* follows VLIW architecture → 8 instructions/cycle

* 32 × 32 bit MUL ⇒ Result either 32 bit nor 64-bit.

ADC Quantization Noise

- In/p signal ⇒ Continous in time or analog wave form.
- This signal is converted into digital → By using ADC
- ADC ⇒ Analog to Digital Conversion.
- The process of converting an analog signal to digital is shown in the fig.

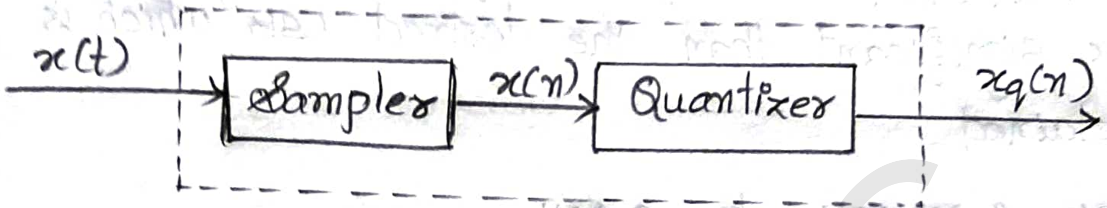


Fig:- Block diagram of A/D Converter.

- Signal $x(t)$ is sampled at regular intervals $t=nT$ where $n=0,1,2$ to create a sequence $x(n)$.
  → This is done by a sampler.
- Numeric equivalent of each sample $x(n)$ → expressed by finite no. of bits ⇒ giving the sequence $x_q(n)$.
- The difference in signal $e(n) = x_q(n) - x(n)$ ⇒ Quantization Noise (or) ADC Quantization noise.
- Assume a sinusoidal signal varying b/w +1 and -1 having a dynamic range 2.
- ADC used to convert the sinusoidal signal → employs $(b+1)$ bits including sign bit.
- No. of levels available for quantizing $x(n) \Rightarrow 2^{b+1}$
- Interval b/w successive levels

$$q = \frac{2}{2^{b+1}} = 2^{-b}$$

q → quantization step size
if b = 3 bits then $q = 2^{-3}$
$q = 0.125$

↳ Common methods of quantization

(1) Truncation

(2) Rounding.

(1) Truncation

Truncation is the process of discarding all bits less significant than the desired LSB which is retained

Eg:- 0.00110011 (8 bits) to 0.0011 (4 bits)

↳ truncate the no. → signal value is approximated by the highest quantization level → not greater than the signal.

(2) Rounding

↳ Rounding of a number of b bits is accomplished by choosing the round result as the b bit no. closest to the original no. unrounded.

↳ Eg: 0.11010 → Rounded to three bits

⇓

0.110 or 0.111

↳ ~~Eg: 0.11011111~~ (~~Round to 8 bits~~)

Error due to truncation and rounding.

- If the quantization method is that of truncation, then the no. is approximated by the nearest level that does not exceed it.
- And in this case, the error ⇒ zero or negative.

$$\text{error} = x_T - x \Rightarrow \text{zero or negative}$$

$x_T \rightarrow$ Truncation value of x.

- Error due to truncation follows the inequality.

$$0 \geq x_T - x \geq -2^{-b} \qquad \text{——— (1)}$$

Eg:- $x = (0.12890625)_{10}$ → Decimal Value

$\Rightarrow (0.00100001)_2$ Binary Equivalent

↳ Truncate to ④ bits

⇓

$$(0.0010)_2 = (0.125)_{10} \Rightarrow x_T$$

$$\text{Error} = x_T - x = 0.125 - 0.12890625$$

$$= -0.00390625 > -2^{-b}$$

$$> -2^{-4} = -0.0625$$

Satisfies the inequality.

- Eqn (1) Satisfies ⇒ Sign-magnitude, 1's Complement, 2's Complement } $x > 0$.

- 2's Complement representation

Magnitude of a negative no.

$$x = 1 - \sum_{i=1}^{b} c_i 2^{-i} \qquad \text{——— (2)}$$

↳ If we truncate the number to 'M' bits

$$x_T = 1 - \sum_{i=1}^{M} c_i 2^{-i} \quad \text{—— (3)}$$

↳ change in magnitude

$$x_T - x = 1 - \sum_{i=1}^{M} c_i 2^{-i} - \left(1 - \sum_{i=1}^{b} c_i 2^{-i}\right)$$

$$x_T - x = \sum_{i=1}^{b} c_i 2^{-i} - \sum_{i=1}^{M} c_i 2^{-i}$$

$$x_T - x = \sum_{i=M}^{b} c_i 2^{-i} \quad \text{—— (4)}$$

$x_T - x \geq 0$ change in magnitude → +ve

Error is -ve & satisfies the inequality

$$0 \geq x_T - x > -2^{-b}.$$

↳ One's Complement

* Magnitude of negative no.

$$x = 1 - \sum_{i=1}^{b} c_i 2^{-i} - 2^{-b} \quad \text{—— (5)}$$

*. No is truncated to M bits.

$$x_T = 1 - \sum_{i=1}^{M} c_i 2^{-i} - 2^{-M} \quad \text{—— (6)}$$

* change in magnitude due to truncation

$$x_T - x = 1 - \sum_{i=1}^{M} c_i 2^{-i} - 2^{-M} - 1 + \sum_{i=1}^{b} c_i 2^{-i} + 2^{-b}$$

$$= \sum_{i=1}^{b} c_i 2^{-i} + 2^{-b} - \sum_{i=1}^{M} c_i 2^{-i} - 2^{-M}$$

$$x_T - x = \sum_{i=M}^{b} c_i 2^{-i}(2^{-M} - 2^{-b}) \quad \text{—— (7)}$$

$$x_T - x < 0$$

* Magnitude decreases with truncation.
* Error is +ve and satisfy the inequality

$$0 \leq x_T - x < 2^{-b}.$$

↳ Sign magnitude.

* In floating point no. System < exponential part, Mantissa part.
* The effect of truncation is visible only in the mantissa.
* Let mantissa is truncated to M bits.

$$x = 2^c M$$

then $x_T = 2^c M_T$ } —— (8)

* Error $e = x_T - x$

$$= 2^c M_T - 2^c M$$

$$e = 2^c (M_T - M). \quad \text{—— (9)}$$

* From eqn (1) ⇒ $0 \geq M_T - M \geq -2^{-b}$

$$\Rightarrow 0 \geq e > -2^{-b} 2^c \quad \text{—— (10)}$$

* Relative error,

$$\mathcal{E} = \frac{x_T - x}{x} = \frac{e}{x} \Rightarrow e = \mathcal{E}x$$

—— (11)

* eqn (10) becomes

$$0 \geq \mathcal{E}x > -2^{-b} 2^c$$

↳ From eqn (8)

$$0 \geq \varepsilon 2^c M > -2^{-b} 2^c$$

$$\Rightarrow 0 \geq \varepsilon M > -2^{-b} \quad \text{———} (12)$$

* If $M = \frac{1}{2}$ → relative error is max.

$$0 \geq \varepsilon > -2.2^{-b} \quad \text{———} (13)$$

* If $M = -\frac{1}{2}$ → relative error range

$$0 \geq \varepsilon > 2.2^{-b} \quad \text{———} (14)$$

* In One's Complement, → error of truncation of +ve values of the mantissa

$$0 \geq M_T - M > -2^{-b}$$

$$0 \geq e > -2^{-b}.2^c \quad \text{———} (15)$$

* with $e = \varepsilon x = \varepsilon 2^c M$ & $M = \frac{1}{2}$

* Max. range of relative error for positive M

$$0 \geq \varepsilon > -2.2^{-b}$$

for -ve M.,

$$0 \leq M_T - M < 2^{-b}$$

$$0 \leq e < 2^c.2^{-b} \quad \text{———} (16)$$

* With $M = -\frac{1}{2}$

Max. range of the relative error for -ve M

$$0 \geq \varepsilon > -2.2^{-b}$$

+ve M

$$0 \leq e < 2^c.2^{-b} \quad \text{———} (17)$$

The probability density fn. P(e) for truncation of fixed point and floating point numbers.

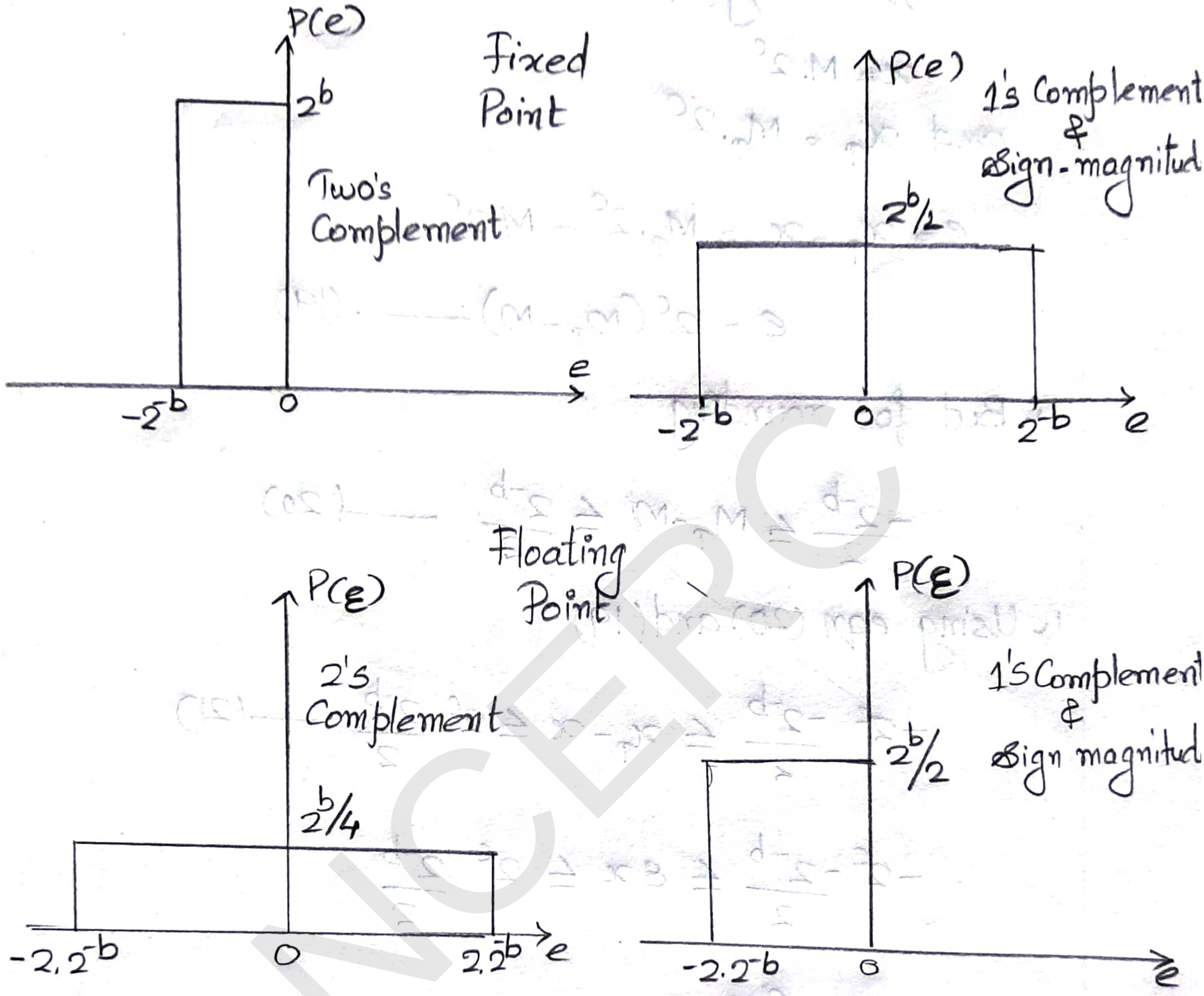


Fig:- Probability density functions P(e) for truncation.

↳ In fixed point no. → error due to rounding a no. to 'b' bits produces an error 'e'

$e = x_T - x$ Satisfies the inequality

$$\frac{-2^{-b}}{2} \leq x_T - x \leq \frac{2^{-b}}{2} \quad \text{——— (18)}$$

↳ If the value lies half way become two levels, it can be approximated to either nearest higher level or by the nearest lower level.

↳ In floating-point arithmetic, only the mantissa is affected by quantization.

$$x = M.2^c$$

and $x_T = M_T.2^c$

$$e = x_T - x = M_T.2^c - M.2^c$$

$$e = 2^c(M_T - M) \quad \text{——— (19)}$$

↳ But for rounding

$$\frac{-2^{-b}}{2} \leq M_T - M \leq \frac{2^{-b}}{2} \quad \text{——— (20)}$$

↳ Using eqn (20) and (19)

$$-2^c \frac{-2^{-b}}{2} \leq x_T - x \leq 2^c \frac{2^{-b}}{2} \quad \text{——— (21)}$$

$$-2^c \frac{-2^{-b}}{2} \leq \varepsilon x \leq 2^c \frac{2^{-b}}{2}$$

$$x = 2^c.M$$

then $-2^c \frac{-2^{-b}}{2} \leq \varepsilon 2^c M \leq 2^c \frac{2^{-b}}{2}$

$$\Rightarrow \quad \frac{-2^{-b}}{2} \leq \varepsilon M \leq \frac{2^{-b}}{2}$$

Mantissa satisfies

$$\frac{1}{2} \leq M < 1$$

↳ If $M = \frac{1}{2}$ → Max. range of relative ~~power~~ error.

$$-2^{-b} \leq \varepsilon \leq 2^{-b}$$

The probability density fn. for rounding

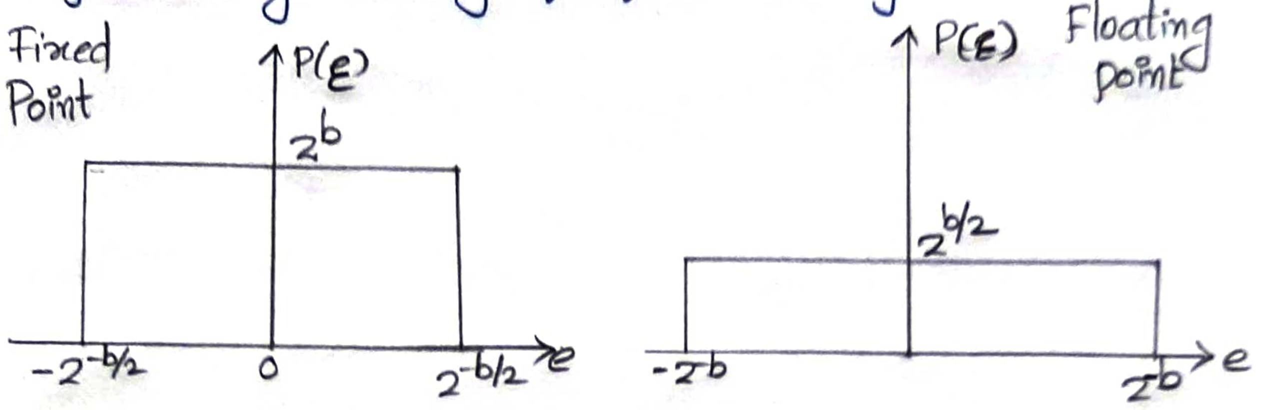


Fig:- Probability density function P(ε) for rounding.

FINITE WORD LENGTH EFFECTS IN DSP SYSTEMS.

Introduction

- DSP algorithms are realized either with special purpose digital hardware or as programs for a general purpose digital Computer.
- In both cases → no.s and Coefficients are stored in finite-length registers.
- Therefore, coefficients and no.s are quantized by truncation or rounding off when they are stored.
- The following errors arise due to quantization of numbers.
  - * Input quantization error
  - * Product quantization error
  - * Coeffecient quantization error.

- Input quantization error.
  - * Conversion of continuous-time input signal into digital Value → produces an error.
  - * This error arises due to the representation of the input signal by a fixed no. of digits in the A/D Conversion process.

- Product quantization error.
  - * errors arise at the o/p of a multiplier.

↳ Coefficient quantization error.

* Filter co-efficients are computed to infinite precision.
* If they are quantized → freq. response of the resulting filter → differ from the desired response → Sometimes filter may fail to meet the desired specifications.
* If the poles of the desired filter are close to the unit circle → filter with quantized coefficients → lie just outside the unit circle → leads to instability.

↳ Other errors arising from quantization are round-off noise and limit cycle oscillations.

Number Representation

↳ a no. N to any desired accuracy by the finite series can be represented as

$$N = \sum_{i=n_1}^{n_2} c_i r^i$$

$r \Rightarrow$ Radix.

Types of Number Representation

↳ Three common forms

* Fixed point representation
* Floating point representation
* Block floating point representation

(1) Fixed Point representation

↳ position of the binary point is fixed.

↳ Binary no. 01.1100 ⇒ 1.75 (decimal)
Integer part ↲ (1) ⟶ fractional part (75)

↳ Three different forms for fixed-point arithmetic.

* Sign-magnitude form.

↳ most significant bit is set to 1 to represent the negative sign.

↳ decimal no. −1.75 ⇒ 11.110000

1.75 ⇒ 01.110000

* One's Complement form.

↳ binary no. → Complement ⇒ 1's Complement.

↳ Addition

Add +0.375 and −0.625 by one's Complement addition

$(0.375)_{10} \Rightarrow (0.011000)_2$

$(0.625)_{10} \Rightarrow (0.101000)_2$

$0.375 \times \frac{2^6}{2^6} \Rightarrow \frac{24}{2^6} = \frac{0.011000}{2^6}$

$0.625 \times \frac{2^6}{2^6} \Rightarrow \frac{40}{2^6} = \frac{0.101000}{2^6}$

↳ 0.375 ⇒ Remove point (dot) and add sign bit in binary.

→ 0011

↳ −0.625 ⇒ Complement (0.625)

→ Remove point and add sign bit (−ve) (1).

+0.625 ⇒ 0.101 $\xrightarrow{\text{Complement}}$ 0.010 $\xrightarrow[\text{Sign bit add. For (−ve)}]{\text{Point Remove}}$ 1010

$$\Rightarrow \begin{array}{r} 0011 \\ 1010 \\ \hline 1101 \end{array} \Big\} \text{Add}$$

No carry.

Since Carry=0 Sum ⇒ Negative

$(1101)_2 = (-0.25)_{10}$

* Two's Complement addition.

↳ represented in 2's complement format → then the addition is performed.

↳ Carry generated in addition → discarded.

↳ C ⇒ 0 → Sum is negative

C ⇒ 1 → Sum is positive

Eg:- Add +0.375 and −0.625 by two's complement addition.

$(0.375)_{10} \longrightarrow (0.011000)_2 \Rightarrow (0.011)_2$

$(0.625)_{10} \longrightarrow (0.101000)_2 \Rightarrow (0.101)_2$

$0.375 \times 2^6 = \frac{24}{2^6} \Rightarrow 0.011000$

$0.625 \times 2^6 = \frac{40}{2^6}$

$(0.375)_{10} \xrightarrow[\text{Add Sign bit}]{\text{Remove Pt}} 0011$

$(-0.625)_{10} \xrightarrow[\text{of binary}]{\text{Complement}} 0.010 \xrightarrow[\text{to LSB}]{\text{Add One}} (0.011)_2 \Rightarrow 0.101000$

$(0.011)_2 \xrightarrow[\text{bit}]{\text{Add Sign}} 1.011 \xrightarrow[\text{dot}]{\text{Remove}} (1011)_2$

$$\Rightarrow \begin{array}{r} 0011 \\ 1011 \\ \hline 1110 \end{array}$$

Carry=0 Sum ⇒ −ve

1110 → Take 2's Complement ⇒ $(-0.010)_2 = (-0.25)_{10}$

(2) Floating Point Arithmetic

↳ Floating Point addition ⇒ no. represent in floating point ~~ad~~ format

↳ addition can be performed only when the exponents of both the no.s are equal.

↳ Exponent of smaller no. is changed to equal the exponent of the larger no. → addition is performed.

Eg Add $(+5)_{10}$ and $(+0.25)_{10}$ by floating point addition. choose 10-bit floating point format with 7-bits for mantissa and 3-bits for exponent.

7 bits ⇒ one bit → Sign
6 bit → no.

3 bits ⇒ 1 bit → Sign
2 → no.

$(+5)_{10} \xrightarrow{\text{binary}} (101)_2$

↓ Add exponent

101.000×2^0 ← expanding

↓ Normalise

decimal point next right should come as 1. So shift to left → 3 places ⇒ 2^3
* shift to right ⇒ 2^{-3}

0.101000×2^3

↓ Convert the exponent to binary.

0.101000×2^{11}

↓ Add sign bit in Mantissa and exponent
+ve no. ⇒ 0
−ve no. ⇒ 1

0.101000×2^{011}

↓ Remove dot

0101000×2^{011} → 3 bit
7 bit

$(+0.25)_{10}$ —Convert to binary→ $(.01)_2$

↓ Add exponent (Compensate to higher value).

$(0.0100000)_2 \times 2^0$

↓ Normalise (decimal point shifting to right → one place.)

$\cdot 100000 \times 2^{-1}$

↓ Convert the exponent to binary.

$\cdot 100000 \times 2^{-01}$

↓ Add sign bit in mantissa and exponent.

0.100000×2^{101}

↓ Remove dot

0100000×2^{101}

Here +5 and +0.25 are not equal. Smaller no. 0.25 is unnormalised to make it equal to that of +5.

$(0.25)_{10} \longrightarrow 0.100000 \times 2^{101}$

↓ Unnormalising.

0.100000×2^{-1}

↓ Unnormalising change to 2^3

0.000010×2^{3} $\quad 2^{-1+4} = 2^3$. (+5 no.)

↓

$0\ 000010 \times 2^{011}$

Add 5 & 0.25

$$\begin{array}{r} 0101000 \times 2^{011} \\ 0000010 \times 2^{011} \\ \hline 0101010 \times 2^{011} \end{array}$$

⇓

$0101010\ 011 \Rightarrow (+5.25)_{10}$

↳ Floating Point Multiplication

* Product is obtained by multiplying the mantissa and adding the exponents.
* Sign bits of mantissa should be added separately to determine the ~~sign~~ sign of product of mantissa.
* For multiplication → exponents used not be same.

Eg:- Multiply $(+5)_{10}$ and $(0.25)_{10}$ by ~~using~~ floating point multiplication. Choose 10-bit floating point format with 7 bits for mantissa and 3 bits for exponent.

Soln. Floating point format.

$(+5)_{10} = 0.101000 \times 2^{3}$

$(+.25)_{10} = 0.100000 \times 2^{-1}$

$5_{10} \times 0.25_{10} = (0+0).(101000 \times 100000) \times 2^{3-1}$

$= 0 \times [0.010100] \times 2^{2}$

$= 0.010100 \times 2^{2}$

↓ Normalise

$.10100 \times 2^{1}$

↓ exponent to binary

0.101000×2^{01}

↓ Sign bit

0.101000×2^{001}

↓ Remove dot

0101000001_2

QUANTISATION OF FILTER COEFFICIENTS.

- FIR and IIR filters → filter coefficients are there.
- In the realisation of FIR and IIR filters in hardware or software, the accuracy with which the filter coefficient can be specified is limited by word length of register used to store coefficients
- Usually the filter coefficients are quantised to the word size of register used to store them either by truncation or rounding.
- Values of poles and zeros → from filter coefficients
- While quantizing → Values of coefficients of filter changes → Location of poles and zeros shifts. → System stability affects.
- Location of poles and zeros of the digital filter directly depends on the value of filter coefficients.
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↳ This will create a deviation in the frequency response of the system. Hence we obtain a filter having a freq. response that is different from the freq. response of filter with unquantised coefficients.

↳ The effects of quantisation of filter coefficients can be minimised by realising a filter with large no. of poles and zeros as an interconnection of second order sections.

↳ Coefficient quantization has less effect in cascade realization when compared to parallel realisation.

Eg:- Consider a second-order filter of having a transfer fn. given by

$$H(z) = \frac{1}{(1-0.9z^{-1})(1-0.8z^{-1})}$$

Find the effect of quantization on pole location of the given system fn. in direct form and cascade form. Take b = (3 bits)

Soln.

$$H(z) = \frac{1}{(1-0.9z^{-1})(1-0.8z^{-1})}$$

$$= \frac{1}{1-1.7z^{-1}+0.72z^{-2}}$$

Quantization of coefficients by truncating to 3-bits
– Direct Method.

$1.7 \xrightarrow{\text{binary}} 1.1011 \xrightarrow[\text{to 3 bits}]{\text{Truncating}} 1.101 \xrightarrow[\text{to decimal}]{\text{Convert}} 2.625$

$0.72 \xrightarrow[\text{to Binary}]{\text{Convert}} 0.1011 \xrightarrow[\text{to 3 bits}]{\text{Truncate}} \rightarrow 0.101 \xrightarrow{\text{deci}} 0.625$

$1.7 \xrightarrow[\text{cation}]{\text{by trun-}} 2.625$

$0.72 \longrightarrow 0.625$

Let $H'(z)$ be transfer fn after quantization.

$$H'(z) = \frac{1}{1 - 2.65z^{-1} + 0.625z^{-2}}$$

It can be seen that there is a large shift in position of poles.

Cascade method

$$H(z) = H_1(z) \cdot H_2(z).$$

$$H_1(z) = \frac{1}{1 - 0.9z^{-1}} \quad \text{and} \quad H_2(z) = \frac{1}{1 - 0.8z^{-1}}$$

Quantise the coefficients by truncating them to 3 bits.

$0.9 \xrightarrow[\text{to Binary}]{\text{Convert}} 0.1110 \xrightarrow[\text{to 3 bits}]{\text{truncate}} 0.111 \xrightarrow[\text{to decimal}]{\text{Convert}} 0.875$

$0.8 \xrightarrow[\text{to Binary}]{\text{Convert}} 0.1100 \xrightarrow[\text{to 3 bits}]{\text{truncate}} 0.110 \xrightarrow[\text{to decimal}]{\text{Convert}} 0.75$

$0.9 \xrightarrow{\text{truncation}} 0.875$

$0.8 \longrightarrow 0.75$

$$H_1(z) = \frac{1}{1-0.875z^{-1}}$$

$$H_2(z) = \frac{1}{1-0.75z^{-1}}$$

Thus it can seen that in cascade form
→ the deviation of poles after quantisation is less compared to the deviation in direct form.

↳ Thus we can say that the effect of quantisation is less in cascade form.

FINITE WORD LENGTH EFFECTS IN FFT ALGORITHMS

ROUND OFF ERRORS.

↳ Quantization errors in FFT algorithms ⇒ Round off errors.

↳ Consider the computation of DFT using radix-2 DIT FFT algorithm for N = 8.

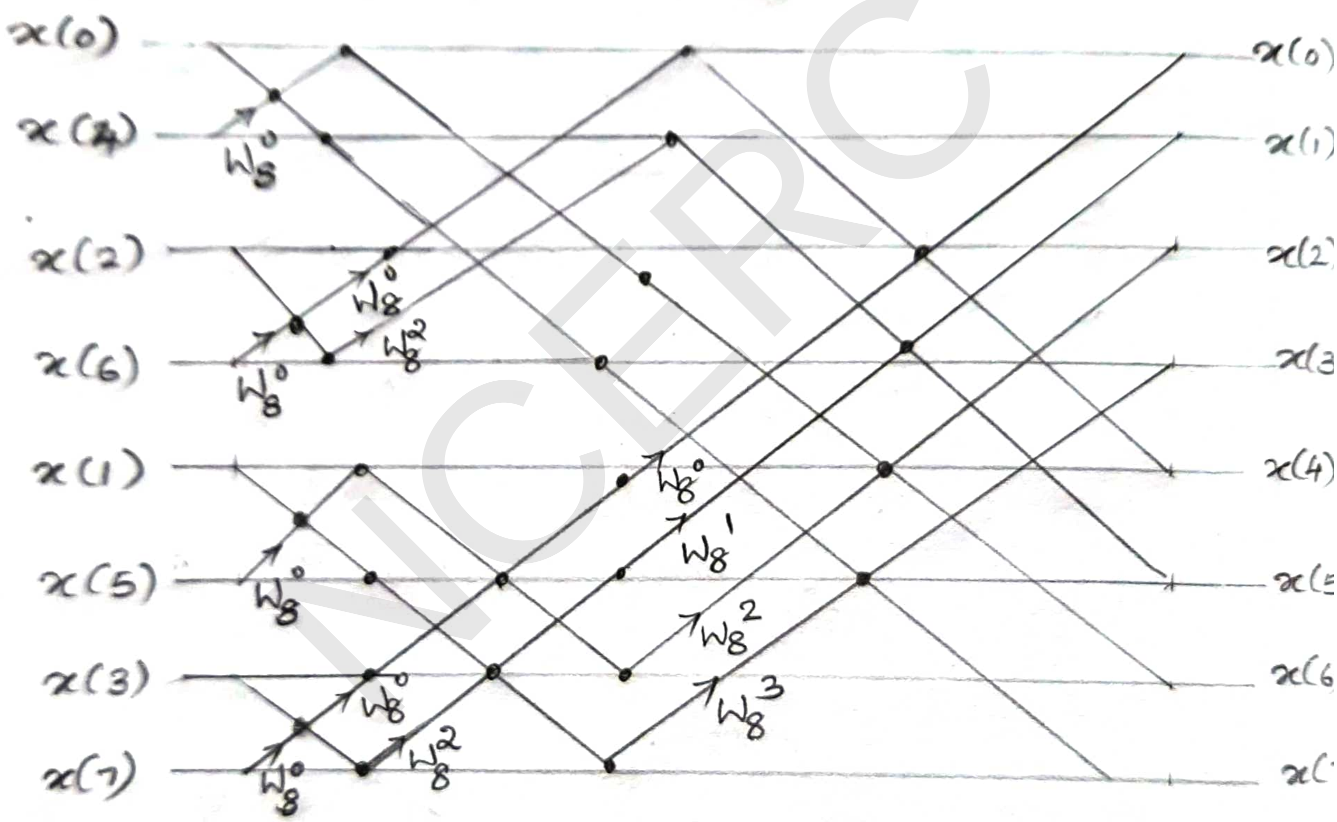

Fig :- Flow Graph for DIT-FFT Algorithm.

↳ DFT is computed ⇒ $M = \log_2 N = 3$ stages.

↳ At each stage a new array of N no. are formed from the previous array by using the basic operation of DIT-FFT algorithm.

$$\left.\begin{aligned} \Rightarrow X_{m+1}(p) &= X_m(p) + W_N^k X_m(q) \\ X_{m+1}(q) &= X_m(p) - W_N^k X_m(q) \end{aligned}\right\} \quad \text{—(1)}$$

where $m, m+1 \rightarrow m^{th}$ array, $(m+1)^{st}$ array.

P and $q \rightarrow$ location of no.s in each array.

↳ Butterfly diagram of eqn (1).

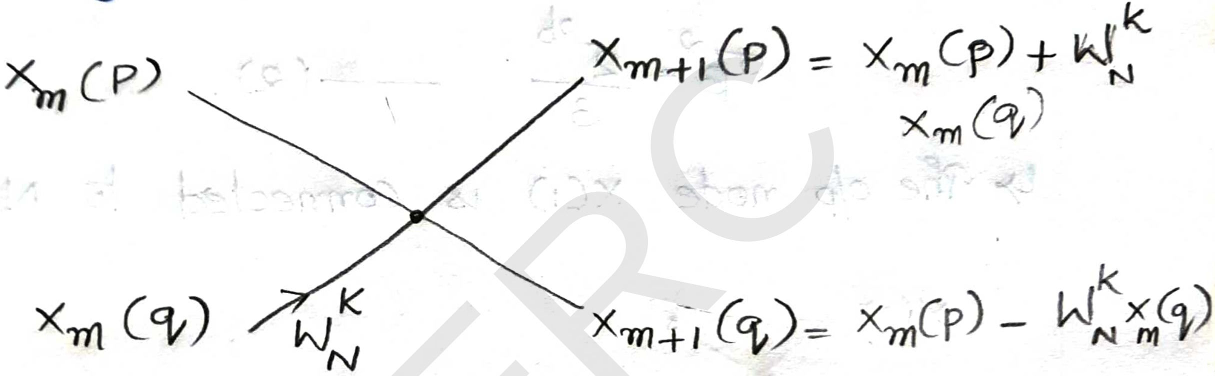

Fig:- Butterfly diagram of eqn (1).

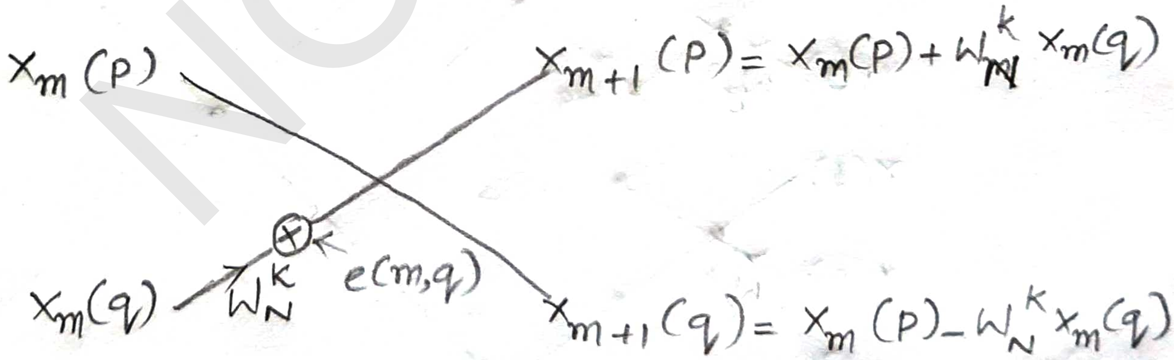

Fig:- Roundoff noise model in DIT - butterfly computation.

* Roundoff noise $\rightarrow$ multiplication.
  ↳ due to which each multiplication is uniformily distributed in amplitude b/w $-2^{-b/2}$ to $2^{-b/2}$
* All noise sources are uncorrelated with each other.
* All noise sources are uncorrelated with in/p and o/p.

↳ i/p data and twiddle factor → Complex

↳ 4 real multiplication are associated for each complex multiply.

↳ Variance of the roundoff error

$$\sigma_\beta^2 = 4\sigma_e^2 = 4 \cdot \frac{2^{-2b}}{12} = \frac{2^{-2b}}{3}$$

$$\sigma_\beta^2 = \frac{2^{-2b}}{3} \quad \text{——} (2).$$

↳ The o/p node X(1) is connected to N−1 = 7 butterflies

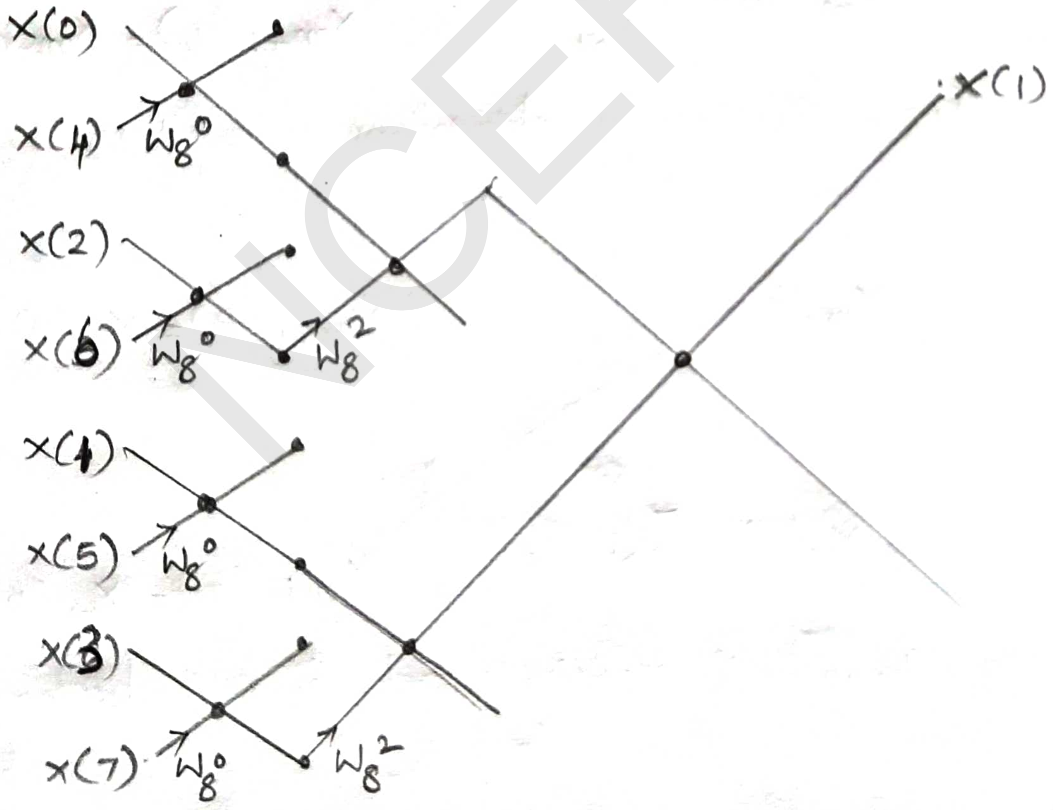

Fig:- Butterflies that effect the computation of X(1).

↳ One noise source → associated with each butterfly.

↳ 7 noise sources affect the computation of $X(1)$.

↳ This is applicable for all output nodes.

↳ In general, $(N-1)$ noise sources propagate to each output node → results in an o/p noise variance given by.

$$\sigma_E^2 = (N-1)\,\sigma_\beta^2 = (N-1)\,\frac{2^{-2b}}{3} \qquad (3)$$

↳ for large value of N.

$$\sigma_E^2 \approx N\,\frac{2^{-2b}}{3} \qquad (4)$$

x ——————— x